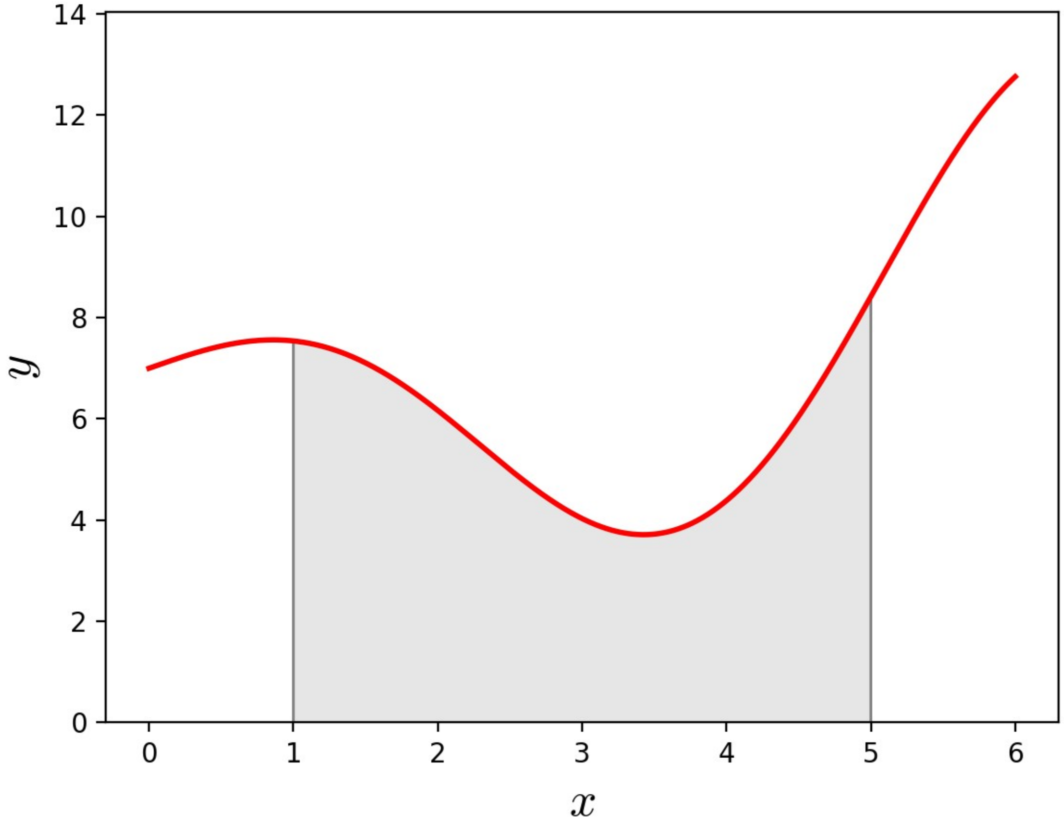
Quadratur-Regeln

$$f(x) = x\cos(x) + 7$$

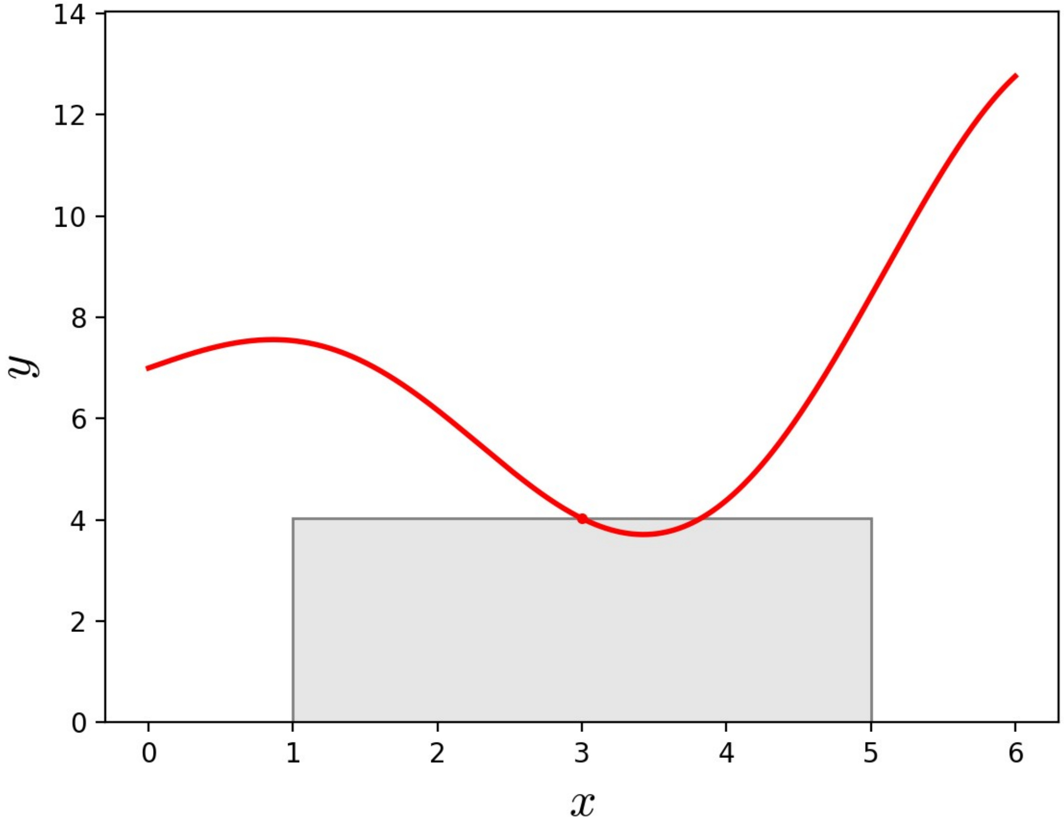


$$I[f] = \int_1^5 f(x)\mathrm{d}x = 22.1073...$$

Bsp.: (9)

Mittelpunktsregel

$$f(x) = x\cos(x) + 7$$

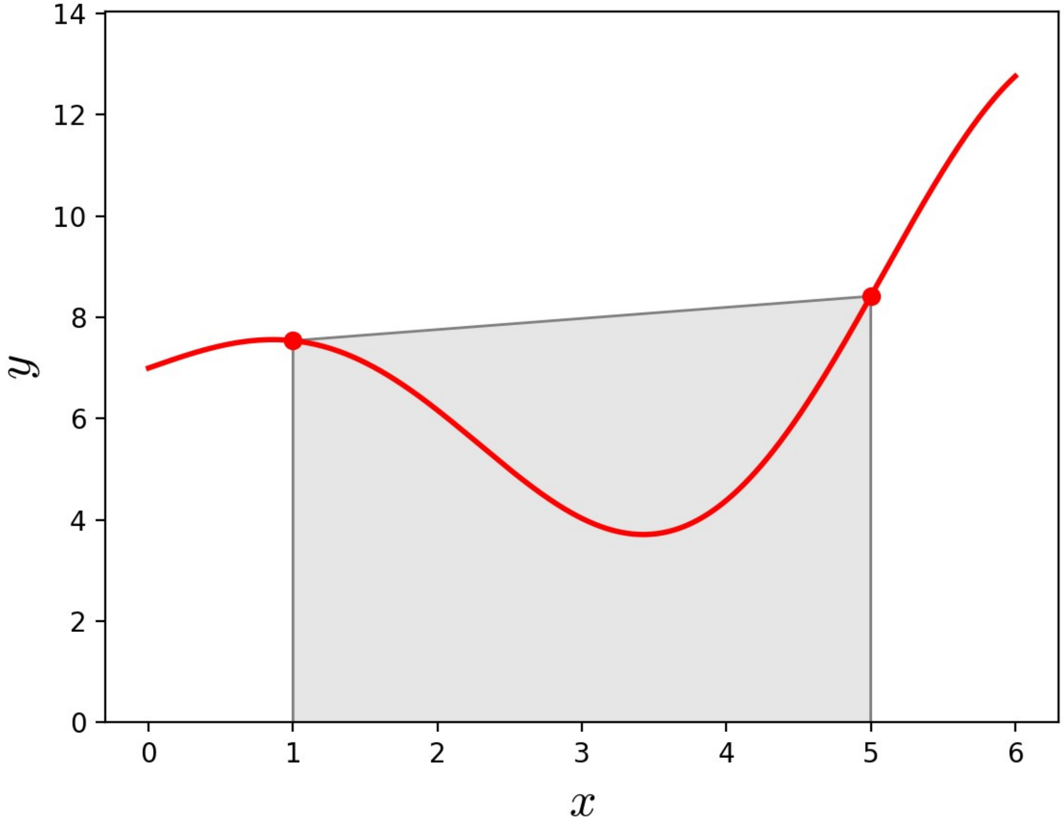


$$I[f] = 22.1073\ldots \quad Q_0[f] = 16.1201\ldots \quad E[f] = 5.9872\ldots$$

Bsp.: (9)

Trapezsregel

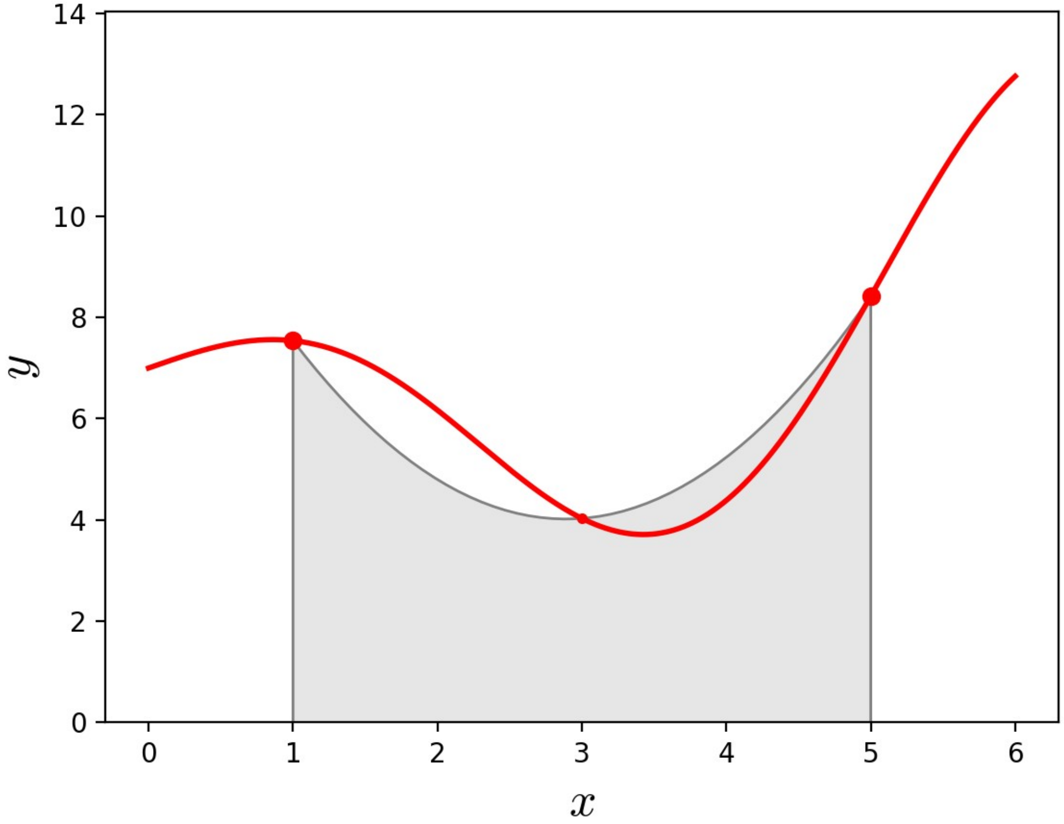
$$f(x) = x\cos(x) + 7$$



$$I[f] = 22.1073\ldots \quad Q_1[f] = 31.9172\ldots \quad E[f] = 9.8100\ldots$$

Simpson-Regel

$$f(x) = x\cos(x) + 7$$

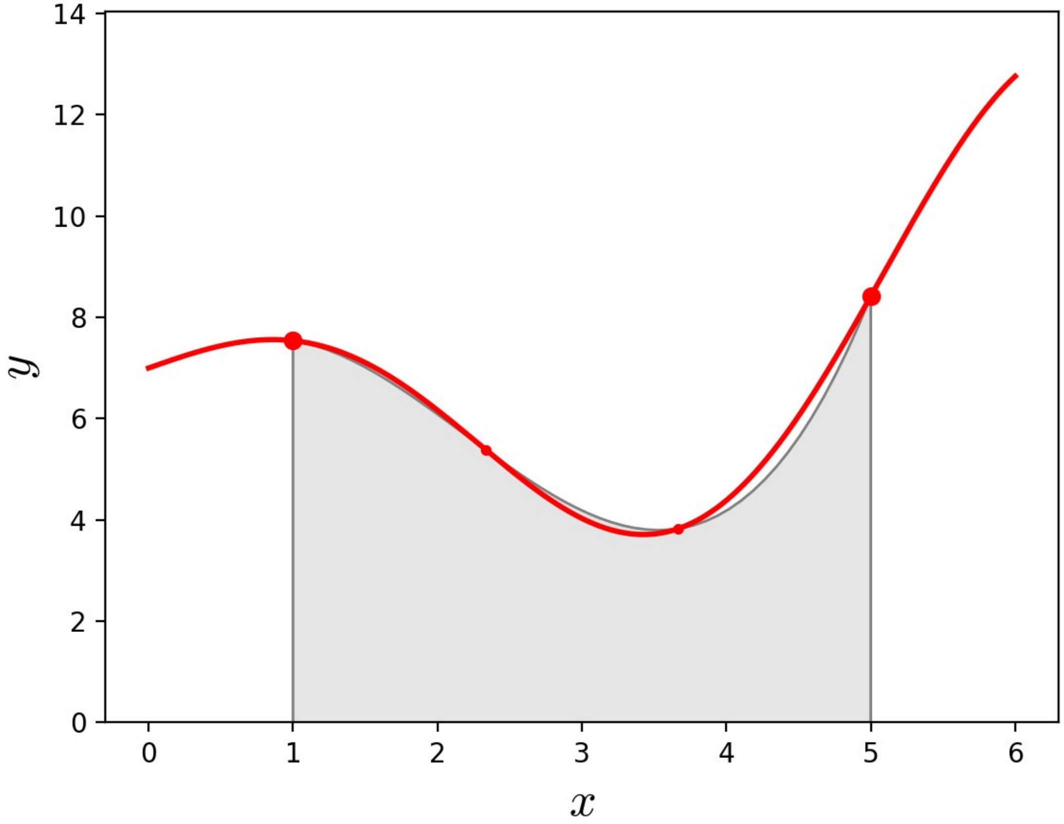


$$I[f] = 22.1073\ldots \quad Q_2[f] = 21.3858\ldots \quad E[f] = 0.7215\ldots$$

Bsp.: (9)

3/8-Regel

$$f(x) = x\cos(x) + 7$$

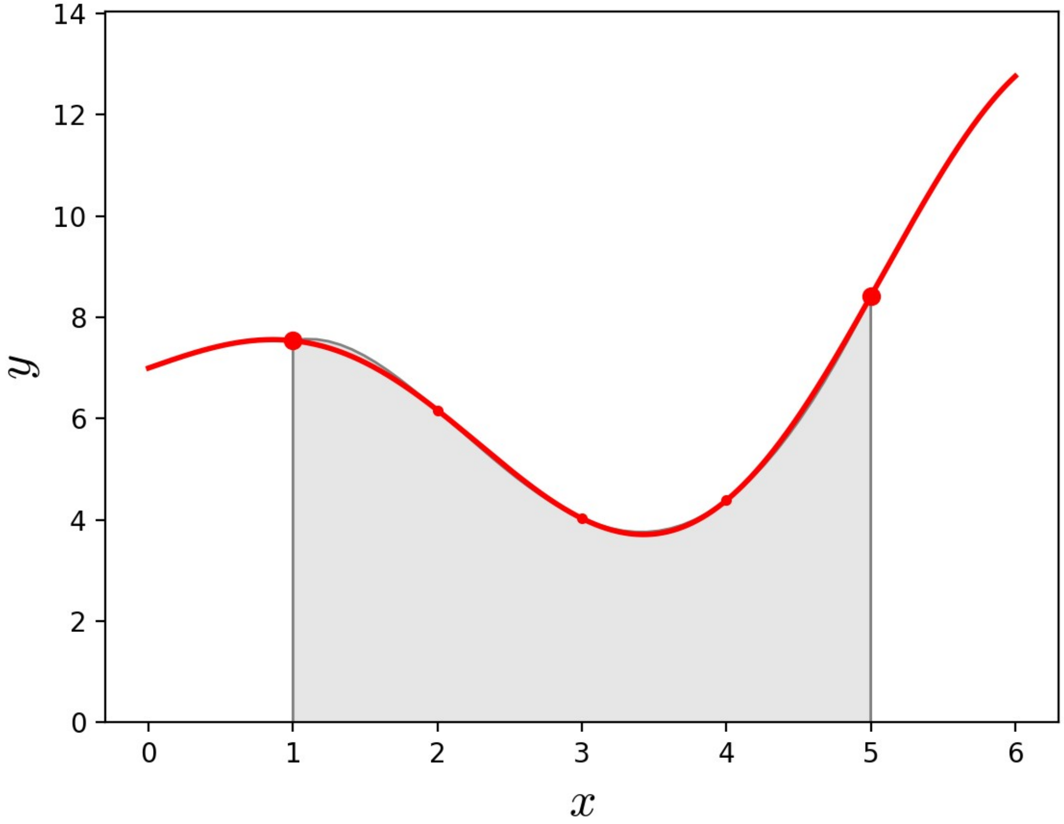


$$I[f] = 22.1073\ldots \quad Q_3[f] = 21.8026\ldots \quad E[f] = 0.3047\ldots$$

Bsp.: (9)

Milne-Regel

$$f(x) = x\cos(x) + 7$$

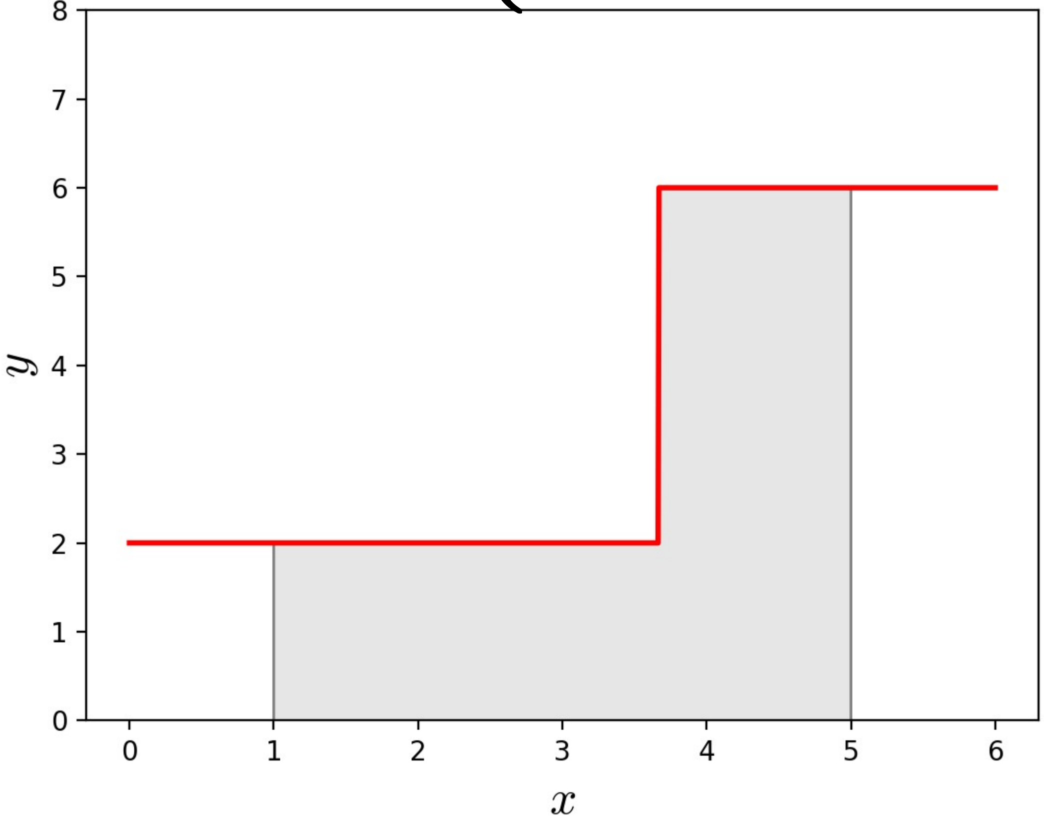


$$I[f] = 22.1073\ldots \quad Q_4[f] = 22.1231\ldots \quad E[f] = 0.0159\ldots$$

Bsp.: (9)

Quadrature

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

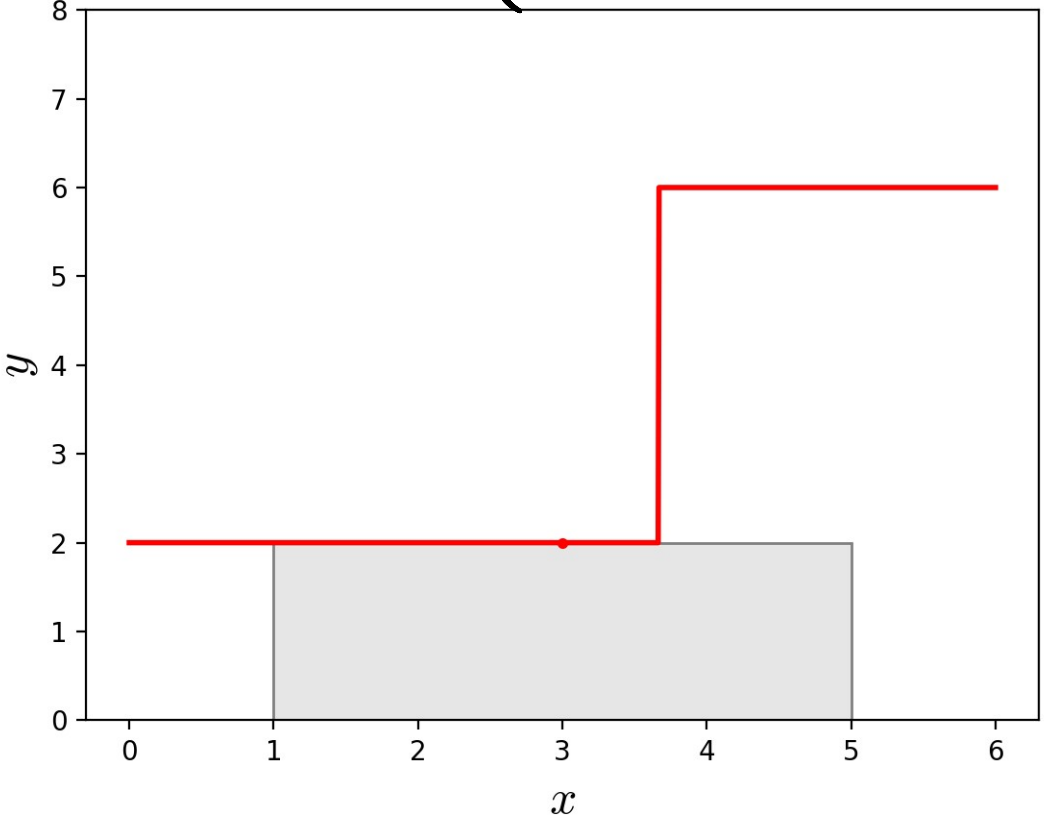


$$I[f] = \int_1^5 f(x)\mathrm{d}x = 15.4336...$$

Bsp.: (9)

Mittelpunktsregel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

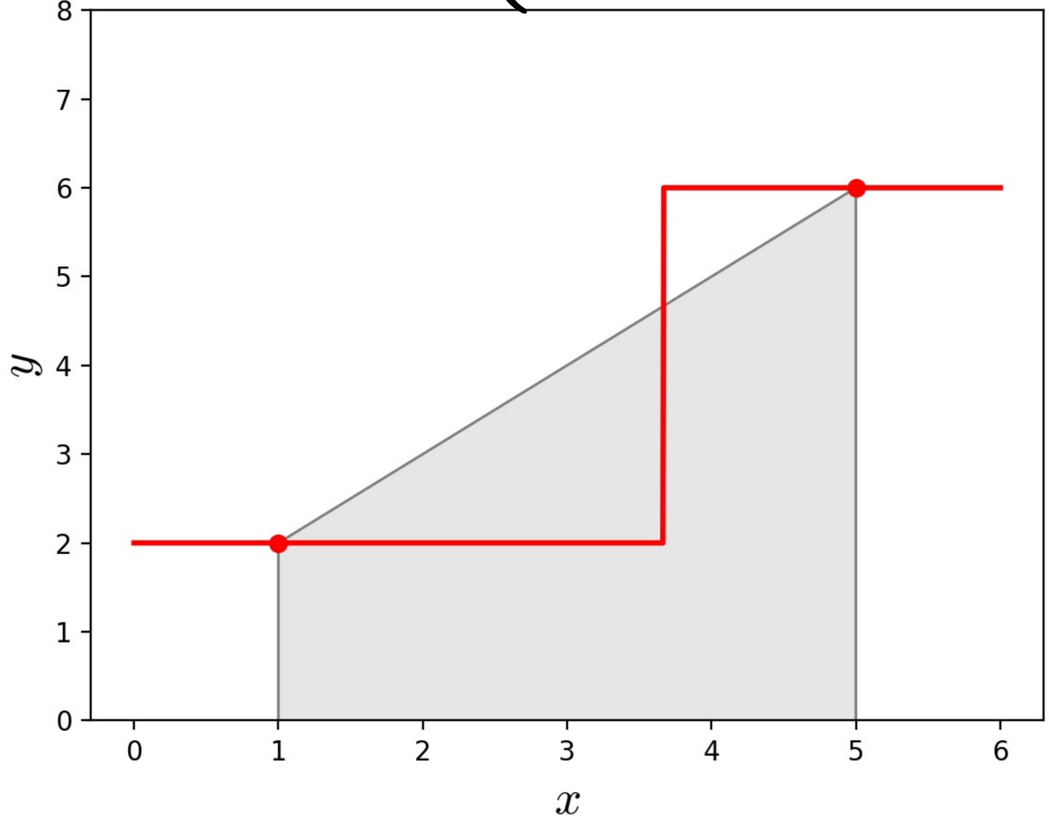


$$I[f] = 15.4336\ldots \quad Q_0[f] = 8 \quad E[f] = 7.4336\ldots$$

Bsp.: (9)

Trapezregel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

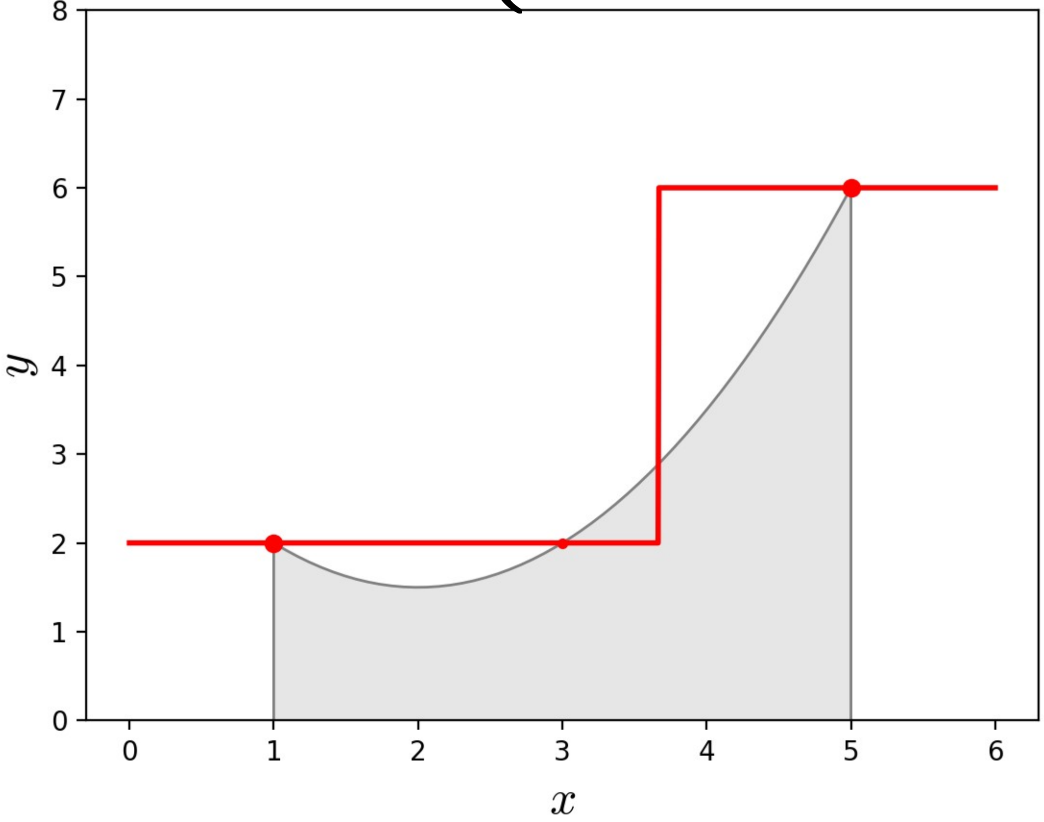


$$I[f] = 15.4336\ldots \quad Q_1[f] = 16 \quad E[f] = 0.5664\ldots$$

Bsp.: (9)

Simpson-Regel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

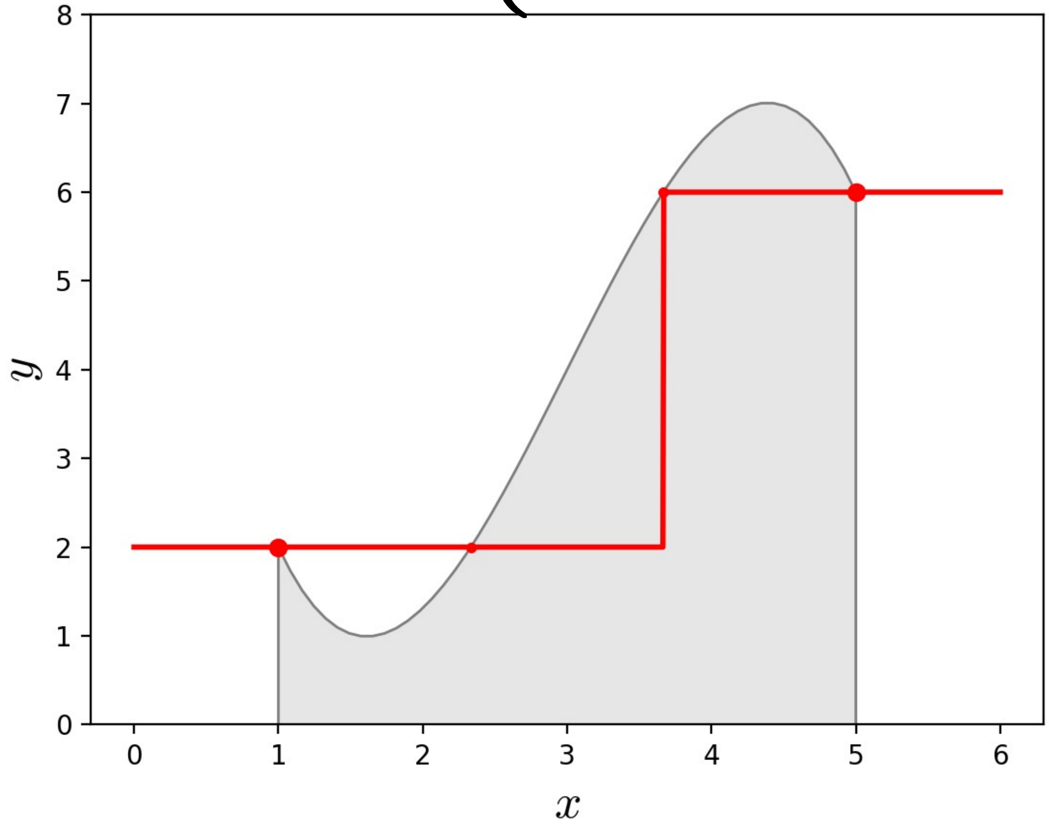


$$I[f] = 15.4336\ldots \quad Q_2[f] = 10.6666\ldots \quad E[f] = 4.7670\ldots$$

Bsp.: (9)

3/8-Regel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

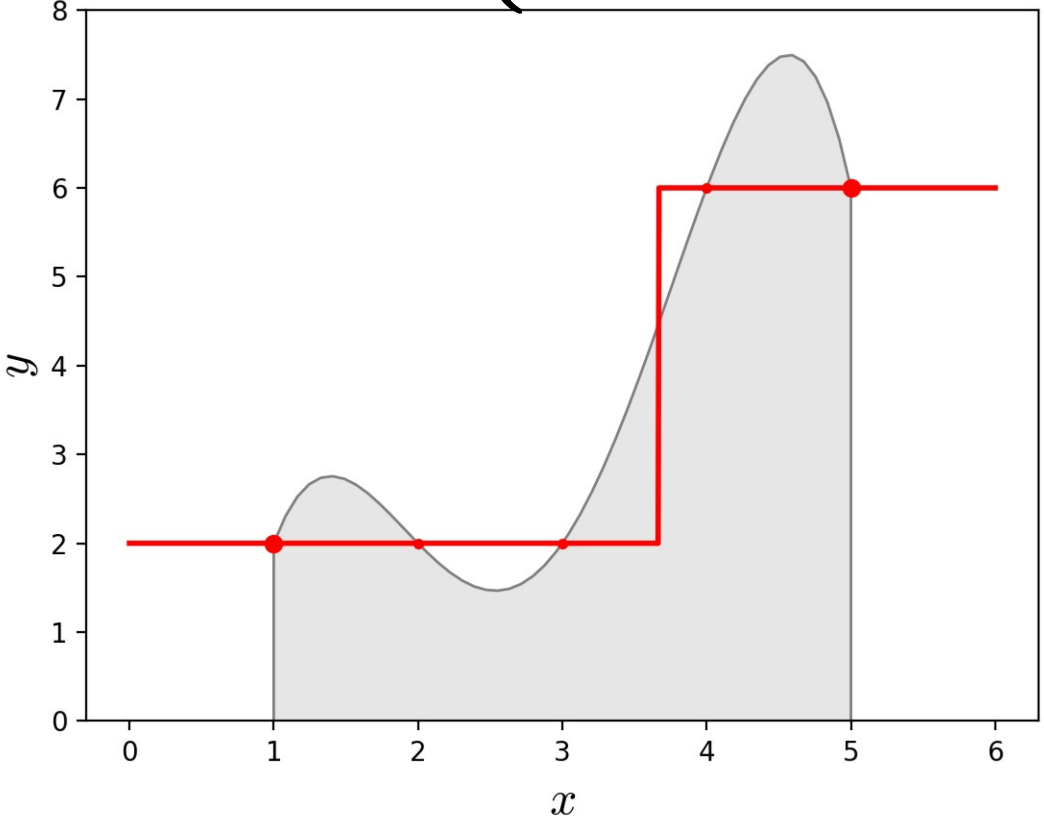


$$I[f] = 15.4336\ldots \quad Q_3[f] = 16 \qquad E[f] = 0.5664\ldots$$

Bsp.: (9)

Milne-Regel

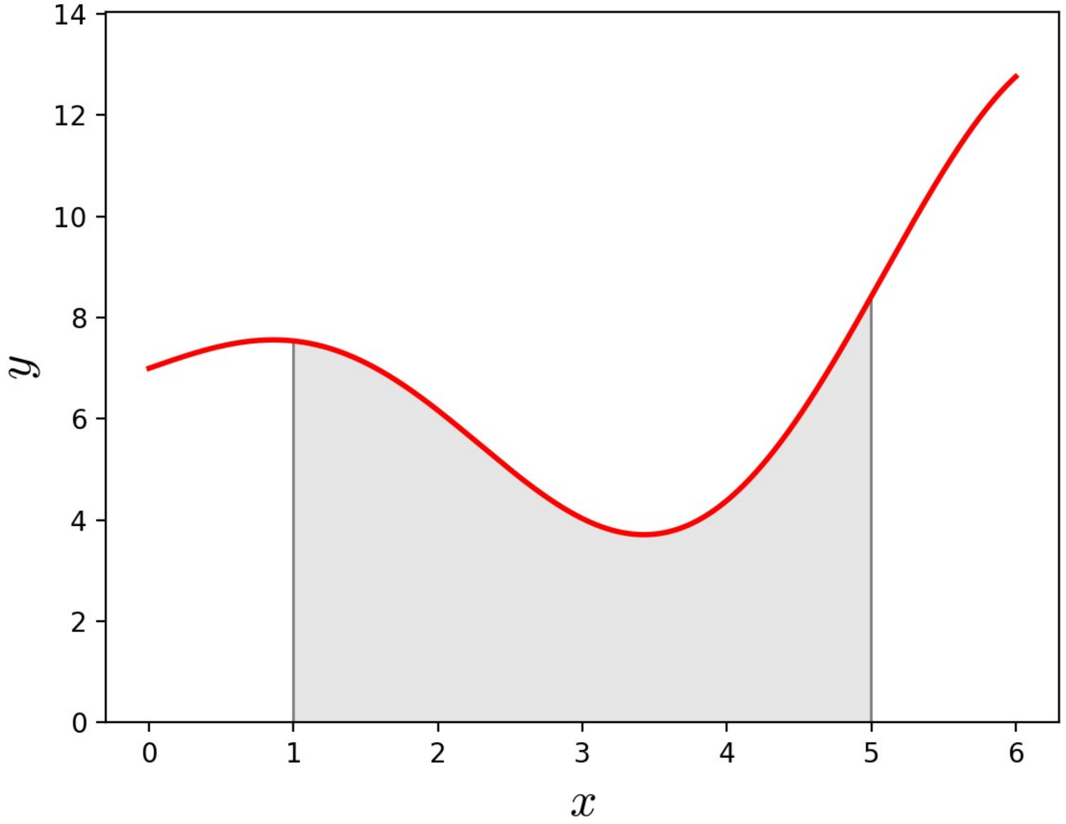
$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$



$$I[f] = 15.4336\ldots \quad Q_4[f] = 14.9333\ldots \quad E[f] = 0.5003\ldots$$

Summierte Quadratur-Regeln

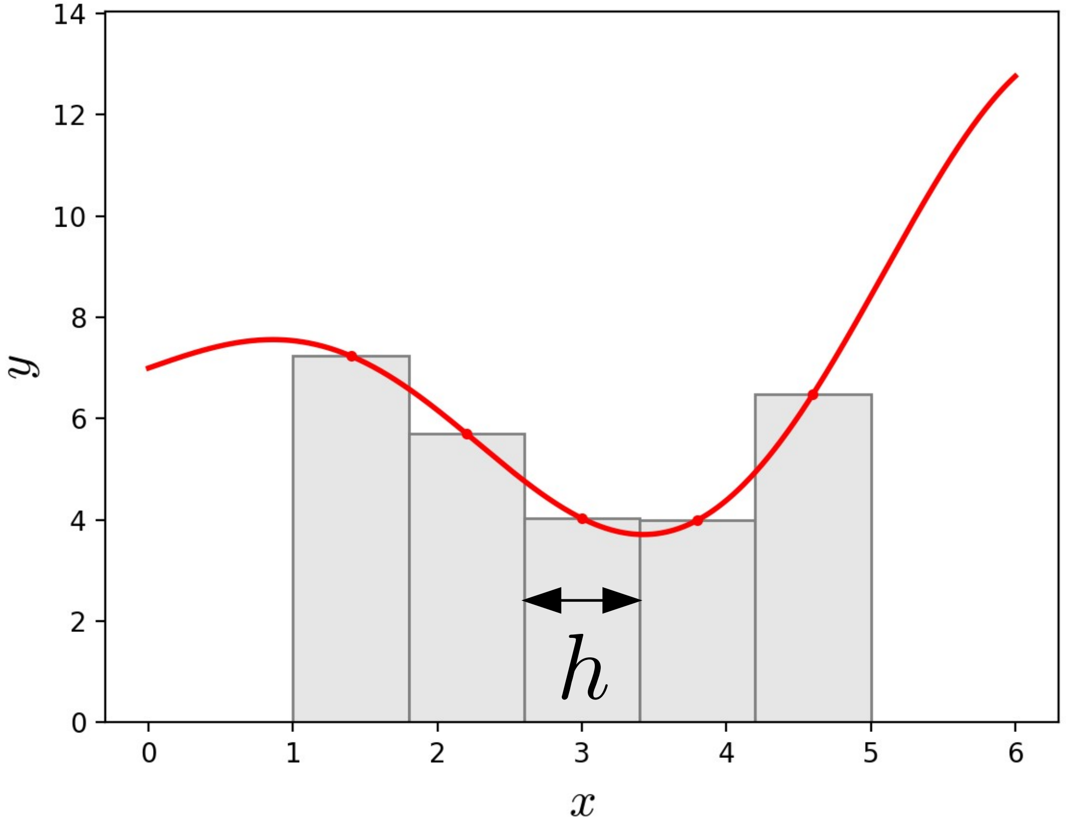
$$f(x) = x\cos(x) + 7$$



$$I[f] = \int_1^5 f(x)\mathrm{d}x = 22.1073...$$

Summierte Mittelpunktsregel

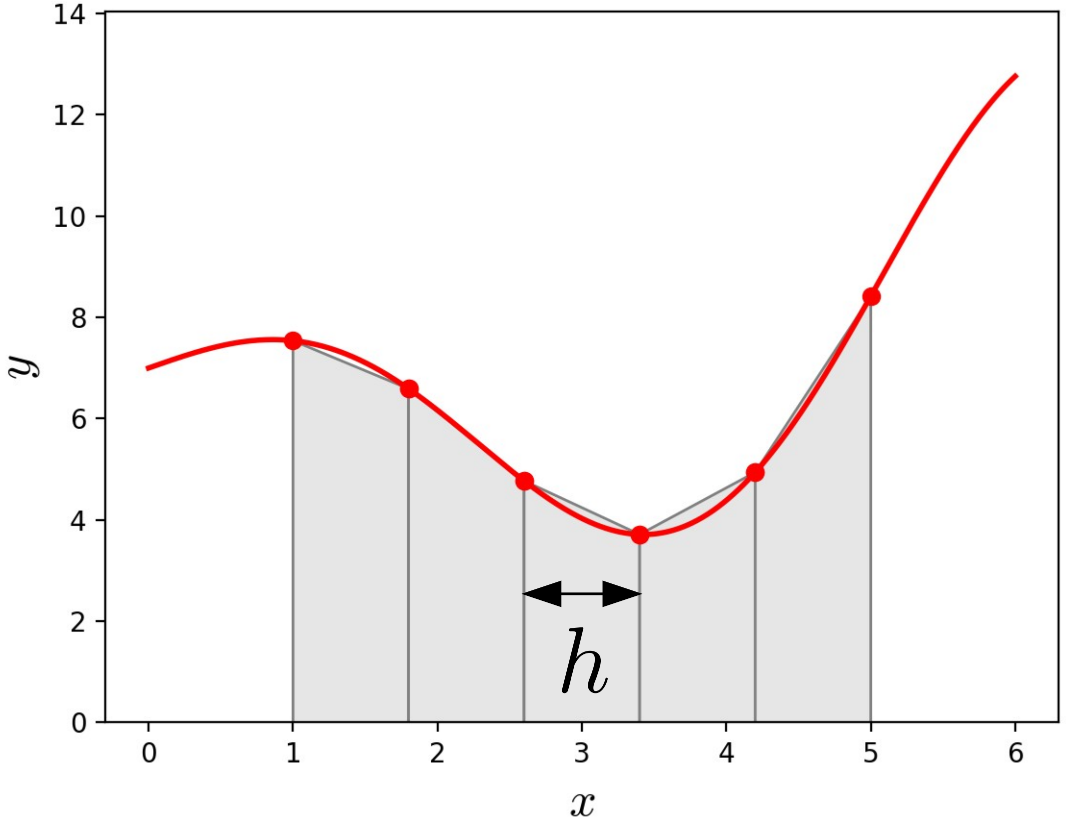
$$f(x) = x\cos(x) + 7$$



$$N = 5$$

Summierte Trapezregel

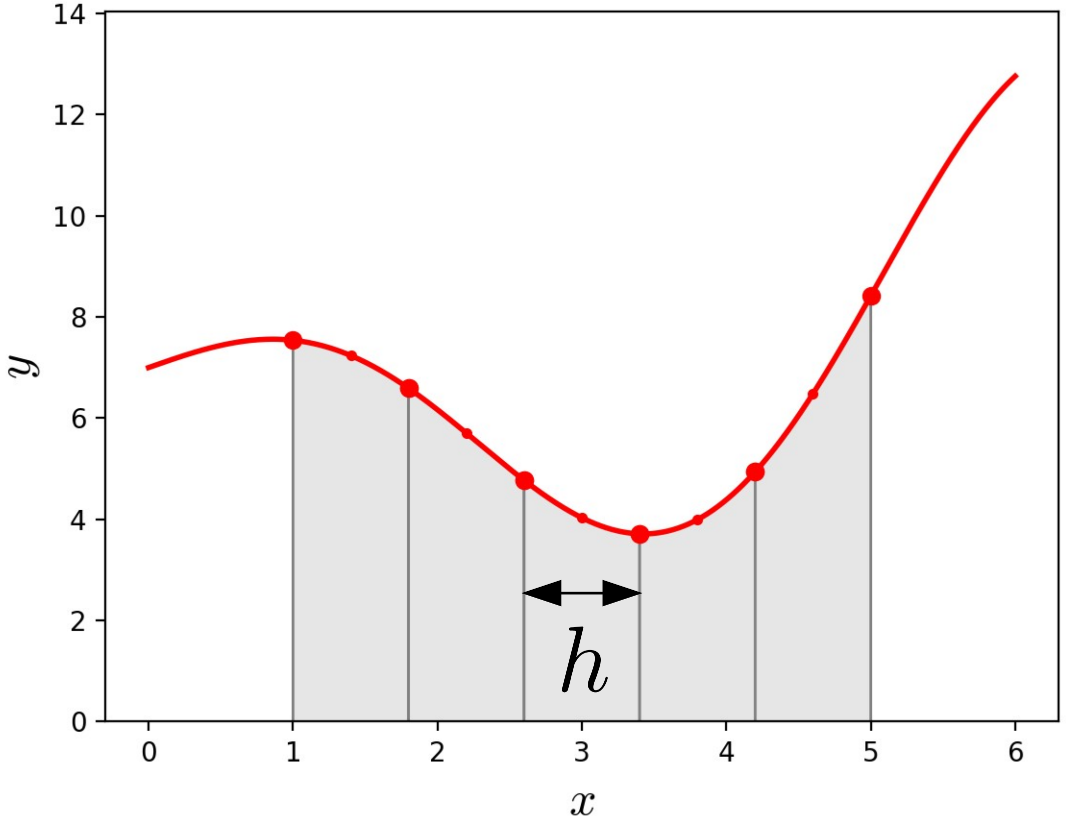
$$f(x) = x\cos(x) + 7$$



$$N = 5$$

Summierte Simpson-Regel

$$f(x) = x\cos(x) + 7$$

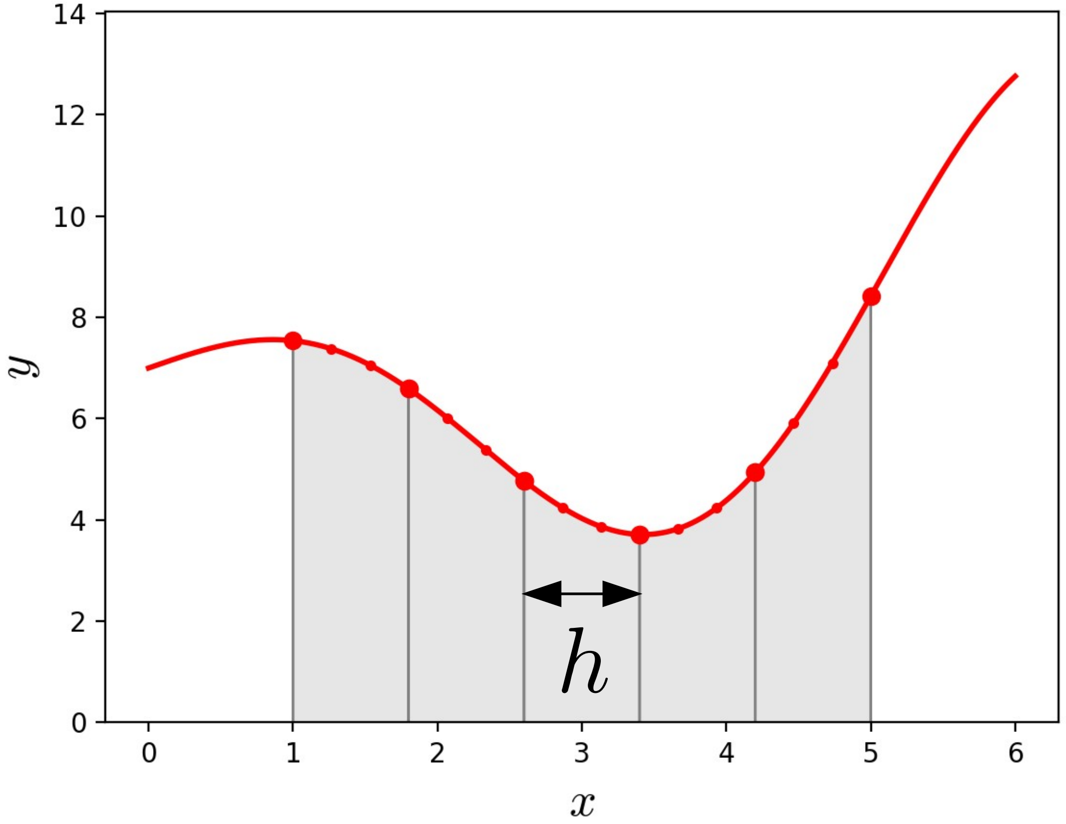


$$N = 5$$

Bsp.: (13)

Summierte 3/8-Regel

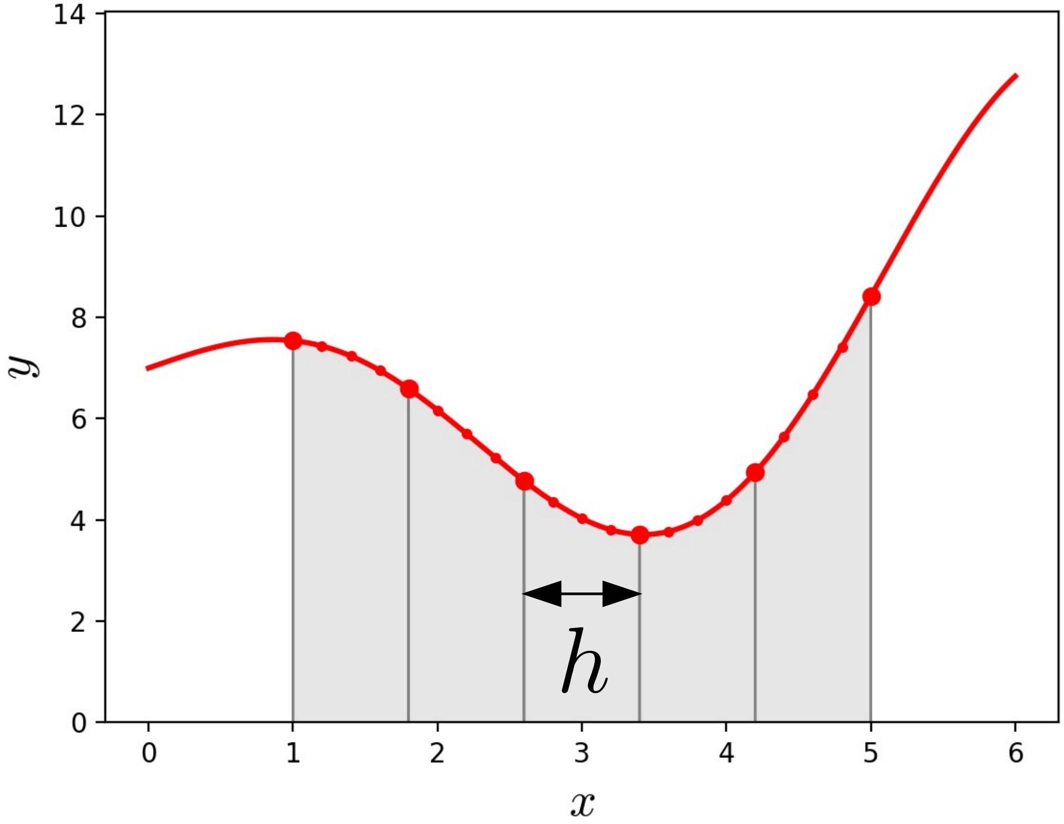
$$f(x) = x\cos(x) + 7$$



$$N = 5$$

Summierte Milne-Regel

$$f(x) = x\cos(x) + 7$$



$$N = 5$$

Summierte Quadratur-Regeln

$$f(x) = x\cos(x) + 7$$

Mittelpunkts — $E_0^N[f]$, Trapez — $E_1^N[f]$, Simpson — $E_2^N[f]$, 3/8 — $E_3^N[f]$, Milne — $E_4^N[f]$

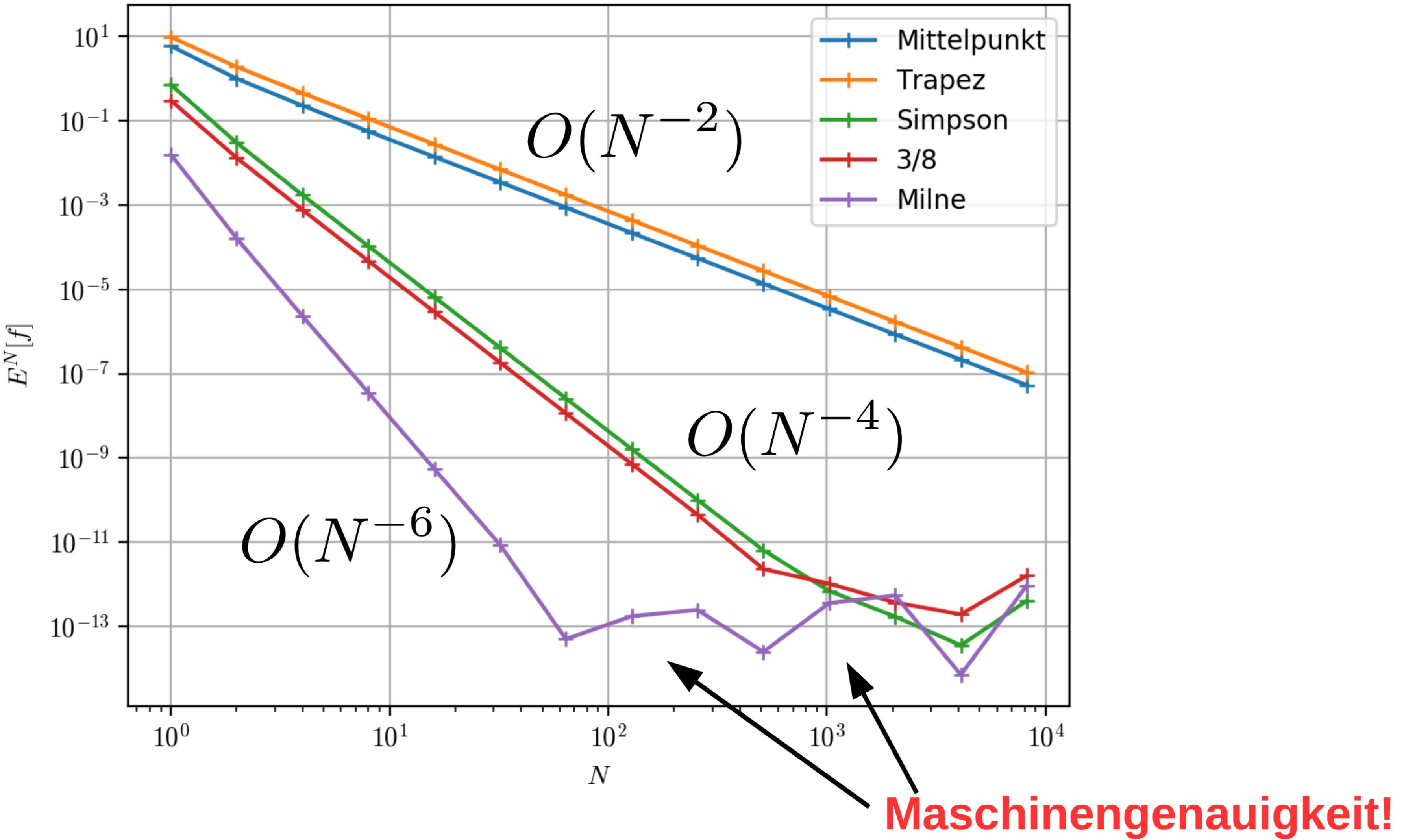
N	h	$E_0^N[f]$	$E_1^N[f]$	$E_2^N[f]$	$E_3^N[f]$	$E_4^N[f]$
1	1.00E+00	5.99E+00	9.81E+00	7.21E-01	3.05E-01	1.59E-02
2	5.00E-01	1.00E+00	1.91E+00	3.02E-02	1.33E-02	1.62E-04
4	2.50E-01	2.30E-01	4.55E-01	1.74E-03	7.69E-04	2.32E-06
8	1.25E-01	5.64E-02	1.12E-01	1.06E-04	4.72E-05	3.54E-08
16	6.25E-02	1.40E-02	2.80E-02	6.61E-06	2.94E-06	5.50E-10
32	3.12E-02	3.50E-03	7.01E-03	4.13E-07	1.83E-07	8.62E-12
64	1.56E-02	8.76E-04	1.75E-03	2.58E-08	1.15E-08	4.97E-14
128	7.81E-03	2.19E-04	4.38E-04	1.61E-09	7.17E-10	1.74E-13

$$h = \frac{b-a}{N}$$

...

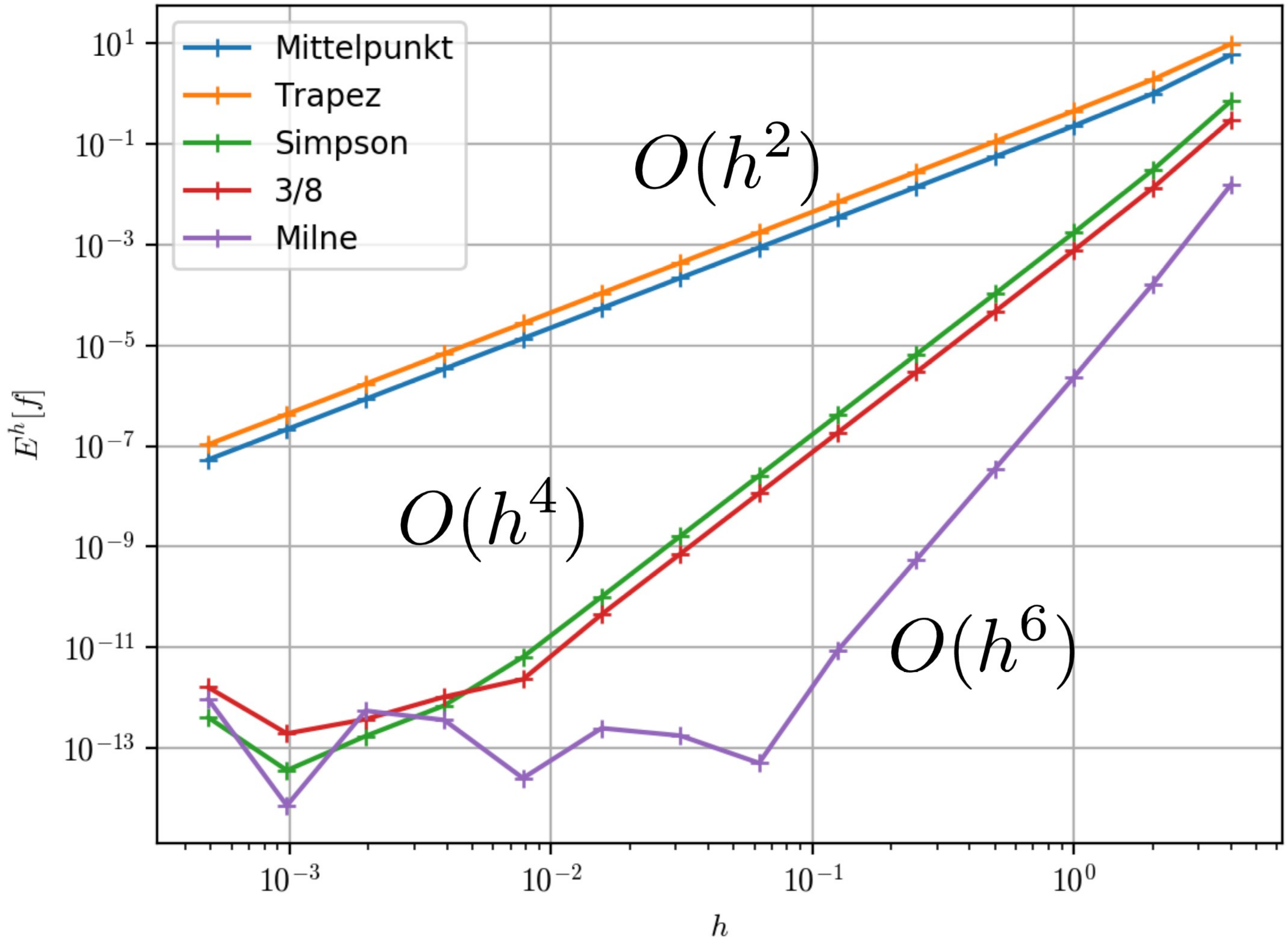
Summierte Quadratur-Regeln

$$f(x) = x\cos(x) + 7$$



Summierte Quadratur-Regeln

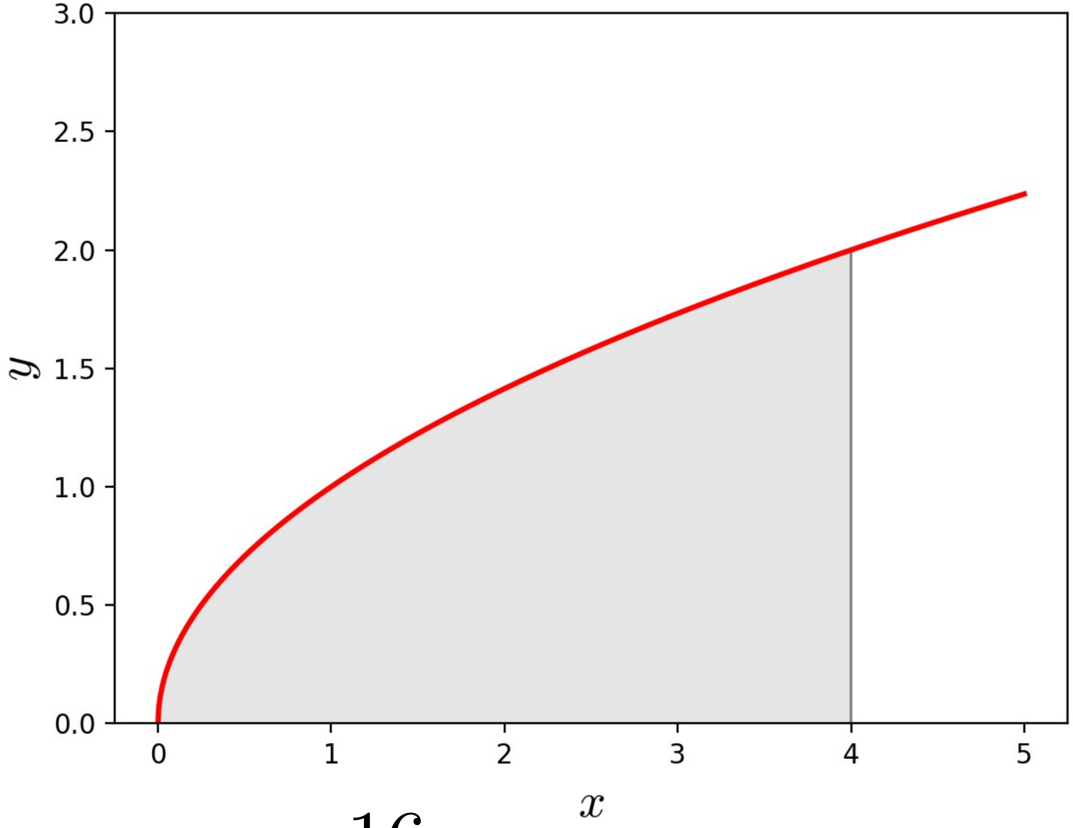
$$f(x) = x\cos(x) + 7$$



Bsp.: (13)

Summierte Quadratur-Regeln

$$f(x) = \sqrt{x}$$



$$I[f] = \int_0^4 f(x)\mathrm{d}x = \frac{16}{3}$$

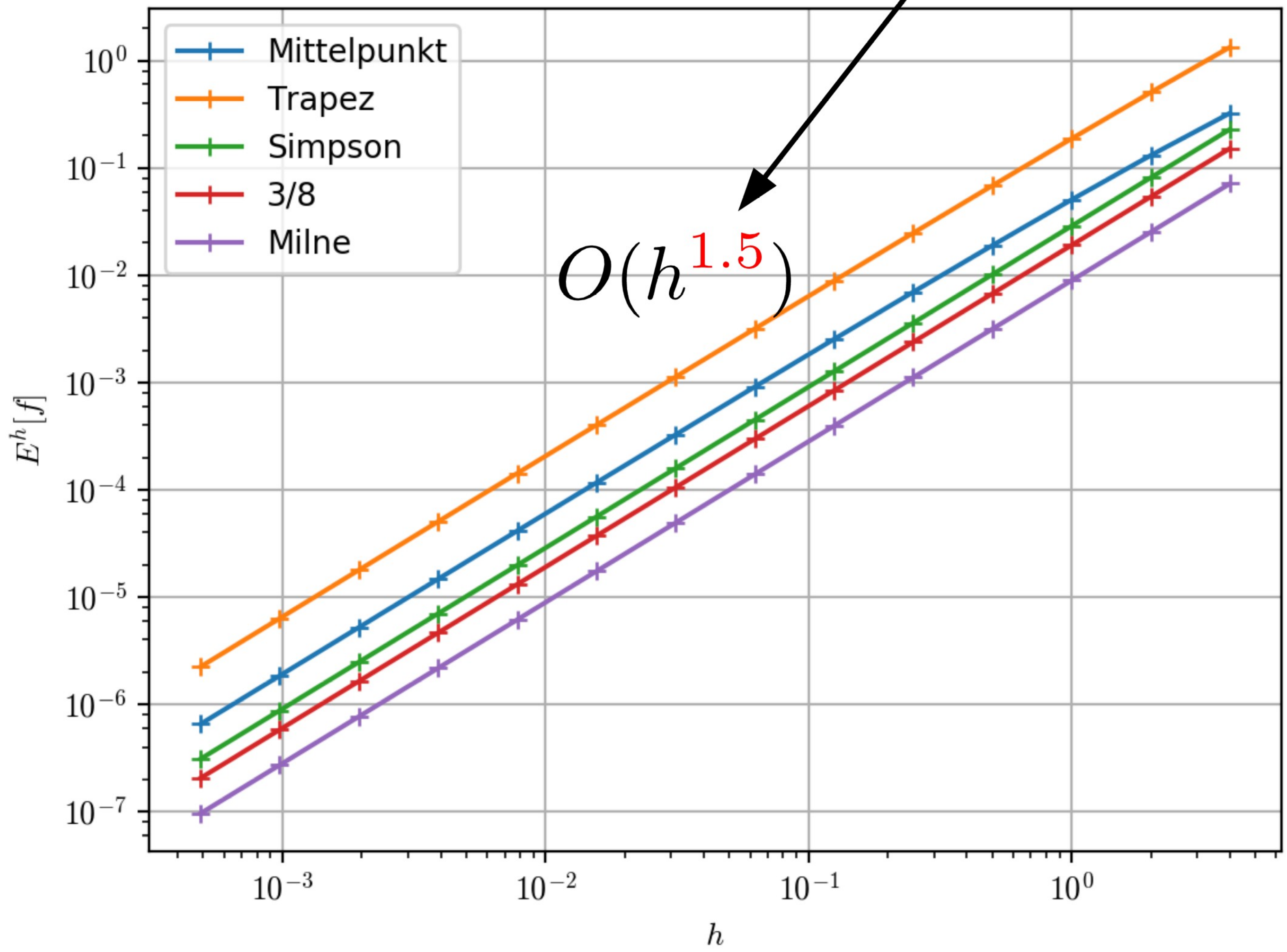
Summierte Quadratur-Regeln

$$f(x) = \sqrt{x}$$

? $f'(x) = \frac{1}{2\sqrt{x}} \xrightarrow{x \to 0} \infty$

...Glattheit!

$O(h^{1.5})$



Gauss(-Legendre) Quadratur

Die (n+1)-Punkte Gauss(-Legendre) Quadratur (GLQ) auf dem RI [-1, 1] ist gegeben durch:

$$G_n[f] = \sum_{j=0}^{n} w_j \cdot f(x_j)$$

wobei die Gauss-Punkte x_j die Nullstellen des (n+1)-ten Legendre-Polynoms $P_{n+1}(x)$ und die Gewichte

$$w_j = \frac{2(1-x_j^2)}{\left((n+1)\,P_n(x_j)\right)^2}, \quad j = 0, 1, \dots, n$$

sind.

Sie hat den grösstmöglichen GG $q = 2n+1$ und damit Ordnung $s = 2n+2$.

Bsp.: (14) 2-Punkte GLQ $(n=1)$

① Berechne $P_{n+1}(x) = P_2(x)$ mit Rekursionsformel

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2-1)$$

② Berechne Nullstellen von $P_2(x)$

$$\frac{1}{2}(3x^2-1) \overset{!}{=} 0$$

$$x_{0,1} = \pm \frac{1}{\sqrt{3}}$$

③ Berechne Gewichte

$$w_0 = \frac{2(1-x_0^2)}{(2\cdot P_1(x_0))^2} = \frac{2(1-1/3)}{(2\cdot(-1/\sqrt{3}))^2}$$

$$= \frac{\cancel{2}\overbrace{(1-1/3)}^{2/3}}{\cancel{4}_{2}\cdot\frac{1}{3}} = 1$$

$$w_1 = 1$$

Also $G_1[f] = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Fehlerschätzer

Bsp.: (15) $I[e^x] = \int_0^1 e^x \, dx = e - 1 = 1.71828...$

TR: $Q_1[e^x] = \frac{1}{2}\left(e^0 + e^1\right) = 1.85914...$

SR: $Q_2[e^x] = \frac{1}{6}\left(e^0 + 4 \cdot e^{1/2} + e^1\right) = 1.71886...$

STR: $Q_1^2[e^x] = \frac{1/2}{2}\left(e^0 + 2 \cdot e^{1/2} + e^1\right) = 1.75393...$

Exakter Fehler : $\left| Q_1[e^x] - I[e^x] \right| = 0.14085...$

Schätzung 1 : $\left| Q_1[e^x] - Q_2[e^x] \right| = 0.14027...$

Schätzung 2 : $\left| Q_1[e^x] - Q_1^2[e^x] \right| = 0.10520...$

Bsp.: (16) TR hat Ordnung $s = 2$ und es ergibt sich für Zahlen aus Bsp. (15) (Schätzung 2):

$$|Q_1[e^x] - I[e^x]| = 0.14085\ldots$$

$$\approx \frac{2^2}{2^2-1} \cdot 0.10520\ldots$$

$$= 0.1402\ldots \quad \checkmark$$

$$|Q_1^2[e^x] - I[e^x]| = 0.0356\ldots$$

$$\approx \frac{1}{2^2-1} \cdot 0.10520\ldots$$

$$= 0.0350\ldots \quad \checkmark$$

Adaptive Quadratur Pseudo-Code

Adaptive Quadratur (~ Pseudo-MATLAB-Code)

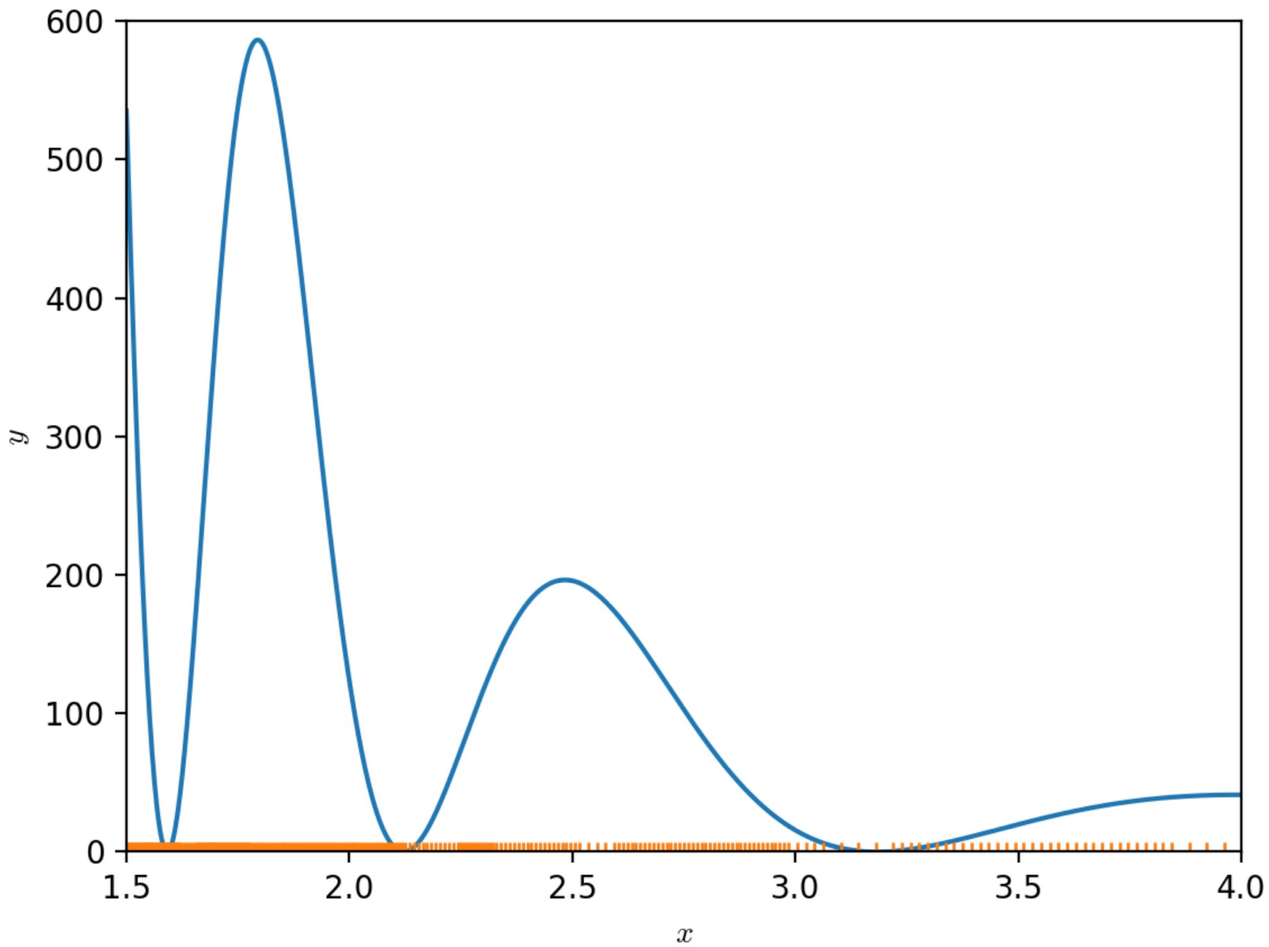
```
function Q = adapt_quad( f, a, b, tol )        % tol: Toleranz
  if  E  <  tol                                 % E: Fehler-Schätzer
      Q = quad( f, a, b )                       % quad: zugrundeliegende QR
  else
      Q1 = adapt_quad( f, a, (a+b)/2, tol/2 )   % REKURSIV!
      Q2 = adapt_quad( f, (a+b)/2, b, tol/2 )   % REKURSIV!
      Q  = Q1 + Q2
  end
```

(zu tol/2:) halbiere Fehler-Toleranz, da nun halbiertes Int.-Intervall!

Bsp.: (17)

Adaptive Quadratur

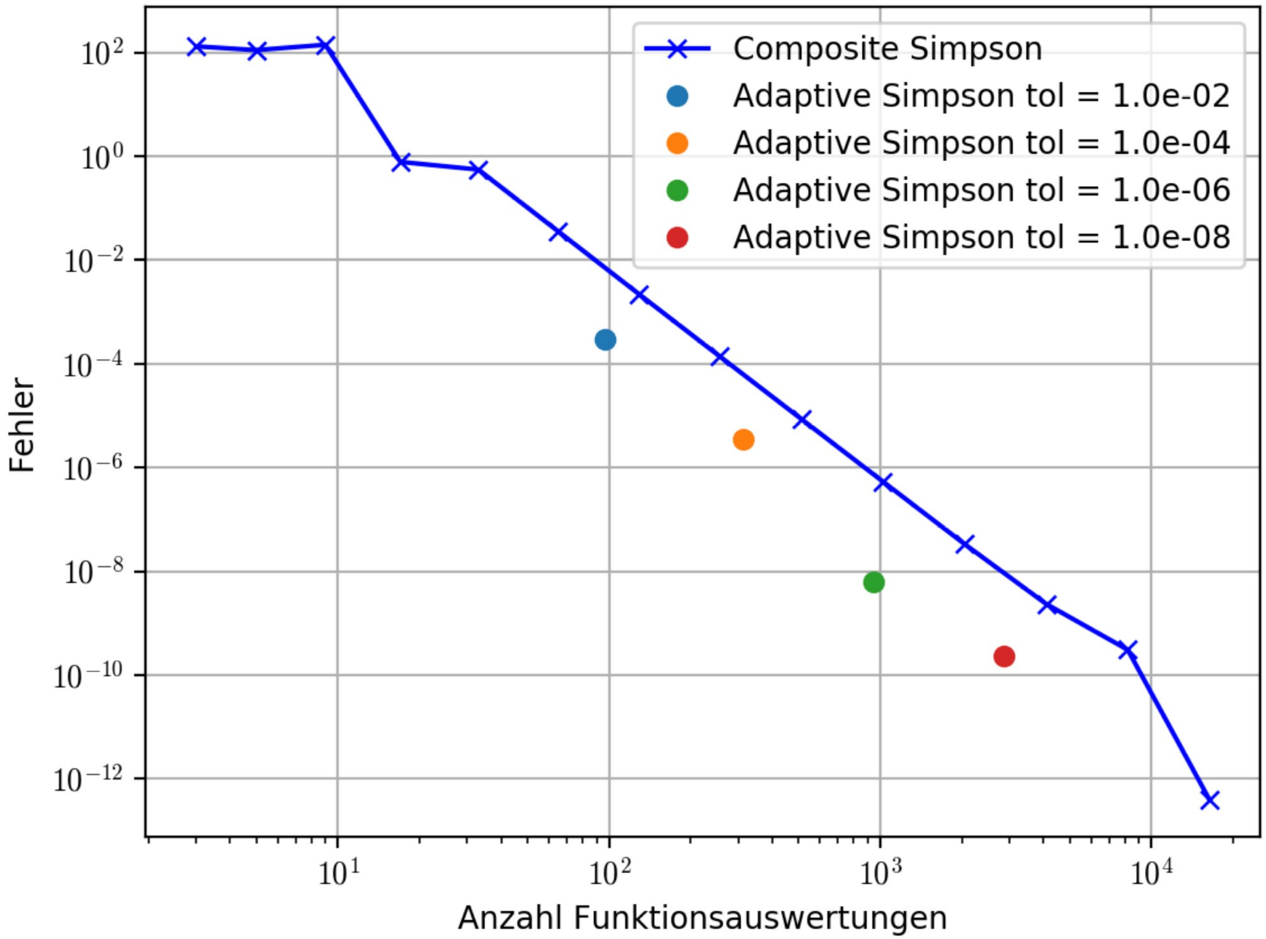
$$I[f] = \int_{3/2}^{4} \left(\frac{200}{2x^3 - x^2} \right) (5 \sin(20/x))^2 \, \mathrm{d}x$$



Bsp.: (17)

Adaptive Quadratur

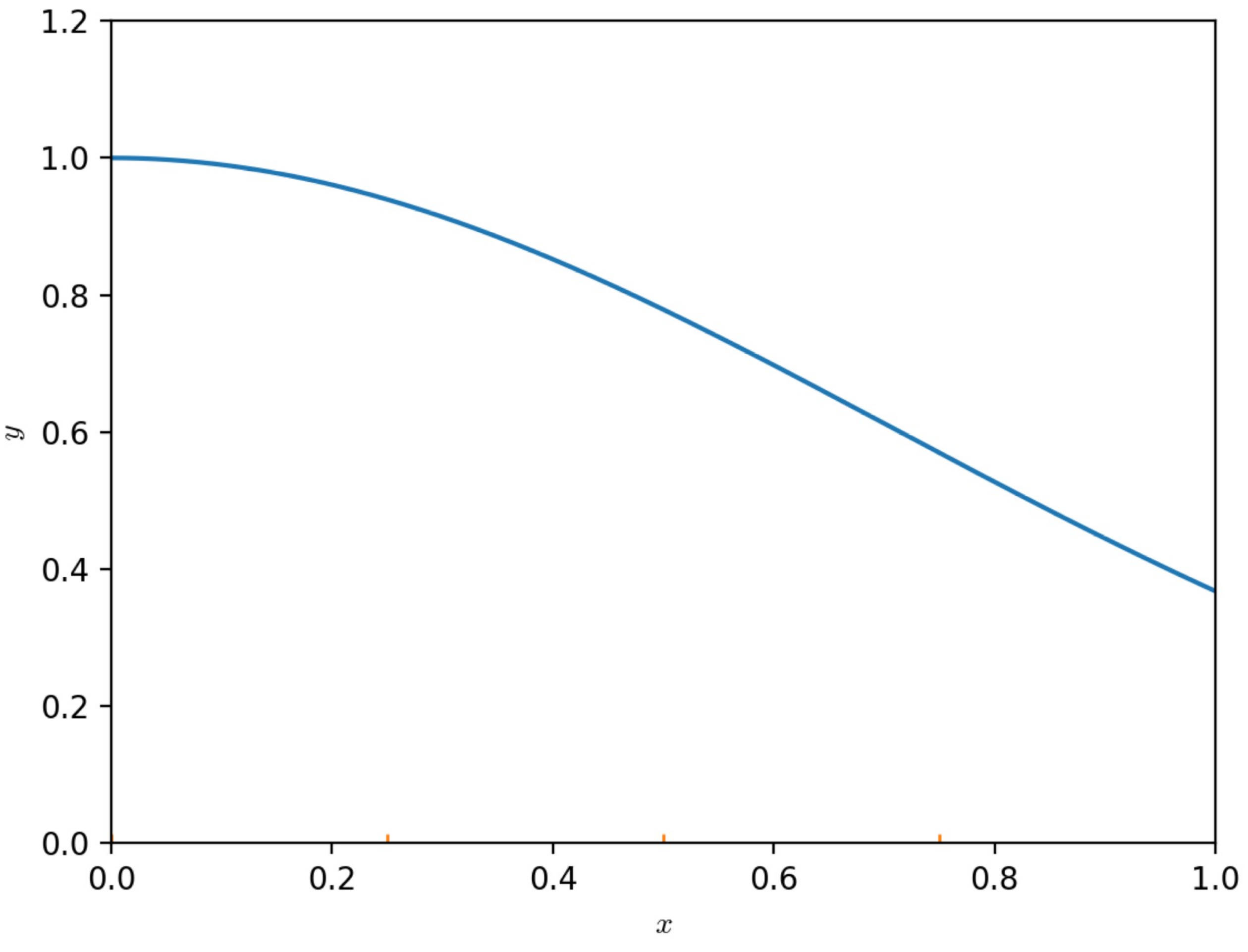
$$I[f] = \int_{3/2}^{4} \left(\frac{200}{2x^3 - x^2} \right) (5 \sin(20/x))^2 \, \mathrm{d}x$$



Bsp.: (17)

Adaptive Quadratur

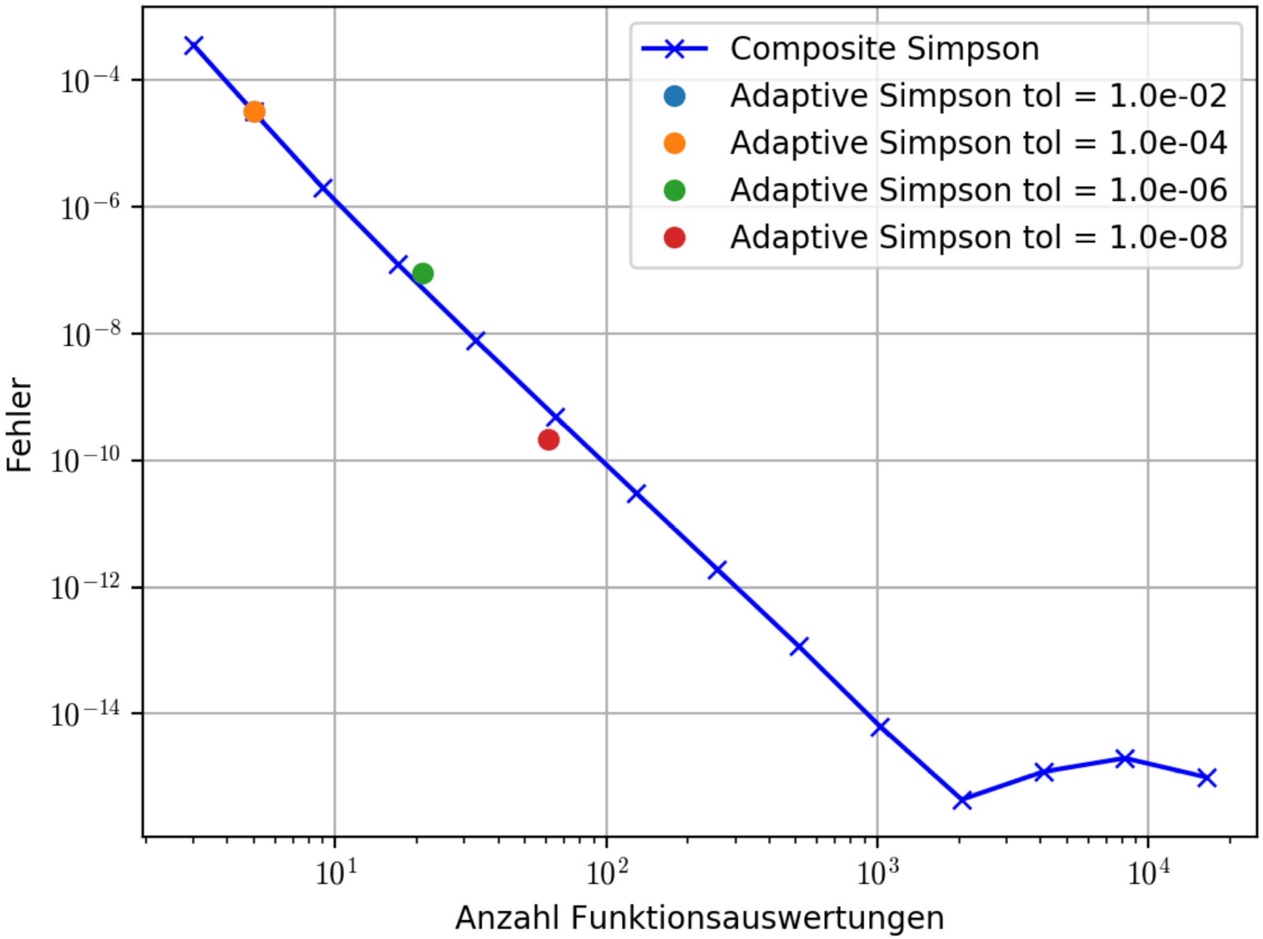
$$I[f] = \int_0^1 e^{-x^2} \mathrm{d}x$$



Bsp.: (17)

Adaptive Quadratur

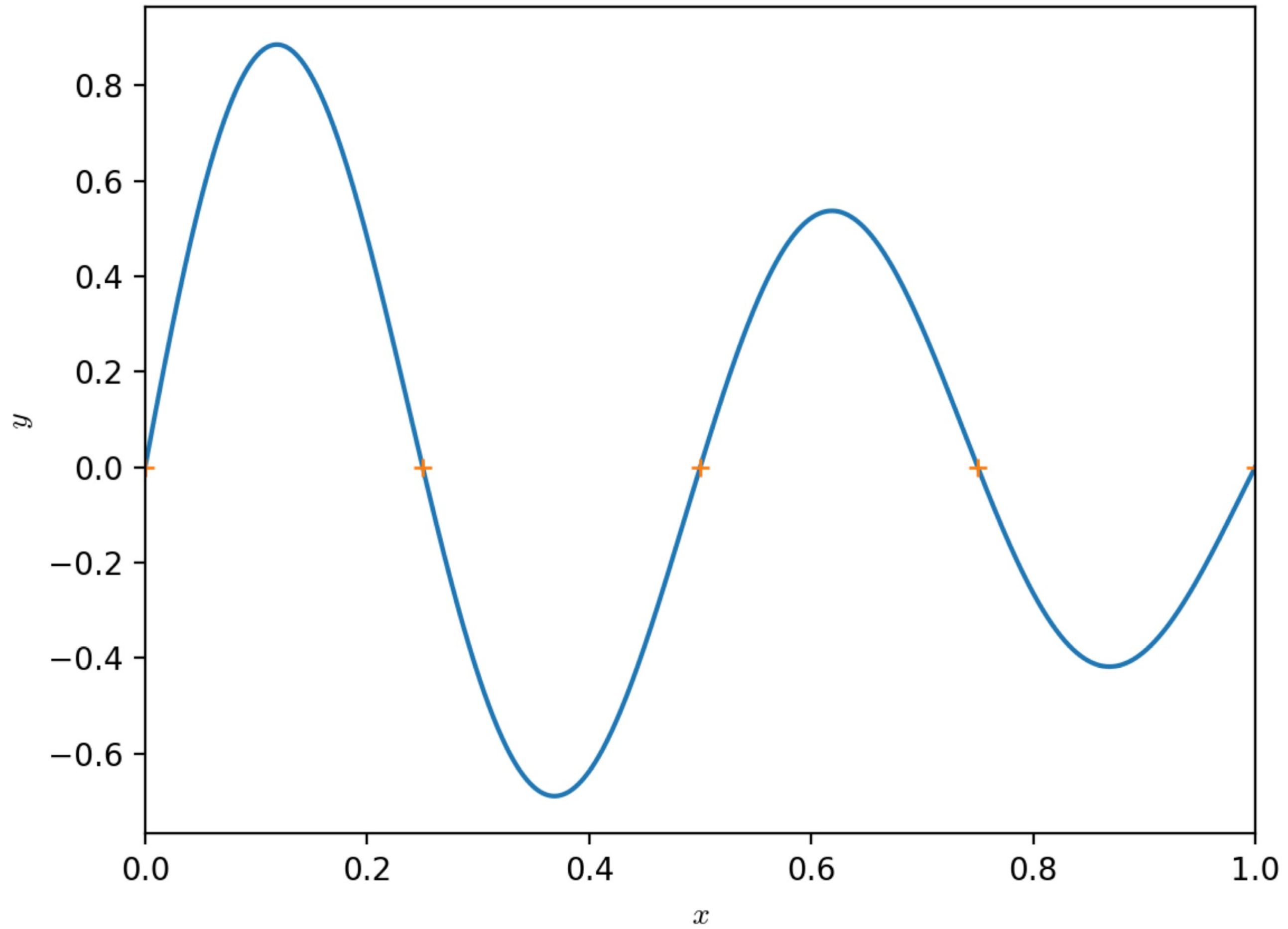
$$I[f] = \int_0^1 e^{-x^2} \mathrm{d}x$$



Bsp.: (17)

Adaptive Quadratur

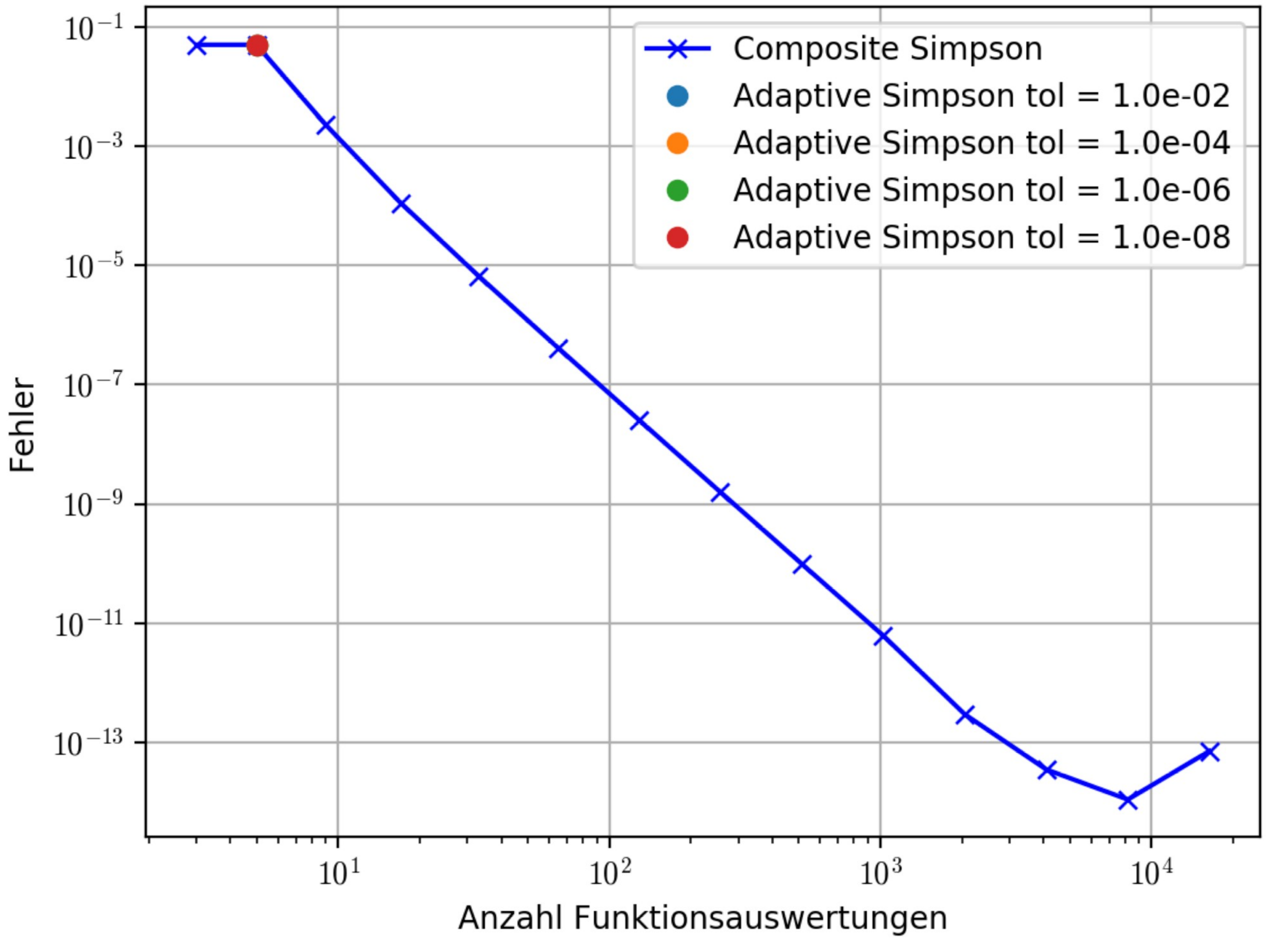
$$I[f] = \int_0^1 e^{-x} \sin(4\pi x) \mathrm{d}x$$



Bsp.: (17)

Adaptive Quadratur

$$I[f] = \int_0^1 e^{-x} \sin(4\pi x) \mathrm{d}x$$



I.8 Zwei-dimensionale Quadratur

$$I[f] = \int_a^b \int_c^d f(x,y)\, dx\, dy$$

$$\approx \sum_{i=0}^{n} w_i^x \cdot \int_c^d f(x_i, y)\, dy$$

Gewichte & Knoten in x-Koordinate

$$\approx \sum_{i=0}^{n} w_i^x \sum_{j=0}^{m} w_j^y \cdot f(x_i, y_j)$$

Gewichte & Knoten in y-Koord.

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} w_i^x \cdot w_j^y \cdot f(x_i, y_j)$$