Exercise Class 2

Abstract Measure Spaces

One might expect that a measure as a generalization of length, area, and volume fulfills the following properties:

i) The measure μ is a non-negative extended real-valued function defined for all subsets of $\mathbb{R}^n$.

ii) The measure is translation invariant, i.e. $\mu(A) = \mu(A + x)$ for $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}$.

iii) Any product of intervals $[a, b]^n$ has measure $(b - a)^n$.

iv) The measure is σ-additive, i.e.

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

for any sequence (A_k) of mutually disjoint subsets of $\mathbb{R}$.

As it turns out, these requirements are incompatible. Thus, we have to lower our expectations. Dropping property ii) or iii) allows us to define a measure on $\mathcal{P}(X)$, however, this notion is in no way a generalization of our geometric intuition for length, area, and volume. Without ii), we obtain e.g. the Dirac measure and without iii), we obtain the trivial measure, i.e., the map which assigns to any set the measure 0. Relaxing the fourth property, that is, not requiring σ-additivity, essentially leads to the theory of Riemann integration, whose limitations put us in this spot in the first place. Instead, we decide to relax property i) and only define the measure for a class of "nice" subsets. We now aim to define a map to pick out a suitable class of subsets (to be called measurable) such that we still have the σ-additivity property.

We will now work in more abstract and general setting in which the Euclidean space $\mathbb{R}^n$ is replaced by a more general space X.

1 Definition of Measures

In order to properly define measure on a general space X, one needs to specify two pieces of data:

i) A collection $\mathcal{A}$ of subsets of X that one is allowed to measure;

ii) A measure $\mu : \mathcal{A} \to [0, \infty]$ which assigns a measure to each set $A \in \mathcal{A}$.

The collection $\mathcal{A}$ should have the structure of a σ-algebra, as it was discussed last week. (For details, see the notes from Exercise Class 1.) We refer to the pair $(X, \mathcal{A})$ as a *measurable space.*

Having now defined the concept of a σ-algebra and a measurable space, we now endow these structures with a measure. What properties should a measure satisfy in general?

Definition 1. A mapping $\mu : \mathcal{A} \to [0, \infty]$ is called a *measure* on X if

i) $\mu(\varnothing) = 0$;

ii) σ-additivity: For all countable collections $(A_k)_k$ of pairwise disjoint sets in $\mathcal{A}$,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

A triplet $(X, \mathcal{A}, \mu)$ is called a *measure space.*

Note that this definition is not the same as in the Lecture Notes. They are not equivalent either! Recall the definition from the lecture.

Definition 2. A mapping $\mu : \mathcal{P}(X) \to [0, \infty]$ is called an *outer measure* on X if

1. $\mu(\varnothing) = 0$;
2. $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ whenever $A \subseteq \bigcup_{k=1}^{\infty} A_k$.

Note that the notion of outer measures are weaker than measures in that they are merely countably subadditive, rather than countably additive. On the other hand, outer measures are able to measure all subsets of X, whereas measures are only defined on a σ-algebra of measurable sets. Why do we not differentiate between these notions in this lecture?

Recall Carathéodory's definition of measurability.

Definition 3 (Carathéodory)**.** Let X be a set with an outer measure μ. We say that $A \subseteq X$ is μ-measurable if for all $B \subseteq X$

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c).$$

Intuitively, μ-measurable sets are the ones that can be used for breaking any other subset (B in the definition) apart into pieces. This way we can compute the measure of the pieces and consider the sum. The naive belief might be that every set is measurable and the measure would satisfy

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \quad \text{for } E_1 \cap E_2 = \varnothing.$$

However, even the most natural of measures (i.e., the Lebesgue measure) fails to satisfy this property, provided one accepts the axiom of choice (think of the Banach–Tarski paradox).

Theorem 4. *Given any outer measure μ on X, the collection of μ-measurable sets of X is a σ-algebra and the restriction of μ to this σ-algebra is a measure.*

We can see that every outer measure gives rise to a proper measure via restriction. Due to that, we do not distinguish between these notions in this lecture.

2 Construction of Measures

A priori, it is not clear how one would construct a measure based on the definition of an (outer) measure. We are, however, able to define finitely additive maps. It is now natural to ask whether we are able to extend this finitely additive map to a measure. In other words, given a finitely additive measure[1] $\tilde{\mu} : \tilde{\mathcal{A}} \to [0, \infty]$ on an algebra $\tilde{\mathcal{A}}$, can we find a refining σ-algebra $\mathcal{A}$ and a σ-additive measure $\mu : \mathcal{A} \to [0, \infty]$ that extends $\tilde{\mu}$?

[1] I am aware that I am being a bit nonchalant with the definitions here.

There is one obvious necessary condition such that $\tilde{\mu}$ can be extended to a measure, namely that $\tilde{\mu}$ already is σ-additive *within* $\tilde{\mathcal{A}}$. That is, suppose that (A_k) is a sequence of mutually disjoint sets in $\tilde{\mathcal{A}}$ such that their union $\bigcup_{k=1}^{\infty} A_k$ is in $\tilde{\mathcal{A}}$ as well, then it is necessary that

$$\tilde{\mu}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \tilde{\mu}(A_k).$$

Using the *Carathéodory-Hahn extension*, we can show that this necessary condition is already sufficient. More precisely:

Definition 5. Let $\tilde{\mathcal{A}} \subseteq \mathcal{P}(X)$ be an algebra. A mapping $\lambda : \tilde{\mathcal{A}} \to [0, \infty]$ is called a *pre-measure* if

i) $\lambda(\varnothing) = 0$;

ii) $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$ for every $A \in \tilde{\mathcal{A}}$ such that

$$A = \bigcup_{k=1}^{\infty} A_k$$

for $A_k \in \tilde{\mathcal{A}}$ mutually disjoint.

Theorem 6 (Carathéodory-Hahn extension)**.** *Let $\lambda : \tilde{\mathcal{A}} \to [0, \infty]$ be a pre-measure on X. For $A \subseteq X$, define*

$$\mu(A) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) : A \subseteq \bigcup_{k=1}^{\infty} A_k, A_k \in \tilde{\mathcal{A}} \right\}.$$

Then μ is an outer measure extending λ and every $A \in \tilde{\mathcal{A}}$ is μ-measurable.
If λ is σ-finite, then the extension is unique.

This theorem allows one to construct a measure by first defining it on a small algebra of sets, where its σ-additivity could be easy to verify. Using this, we are now able to construct a measure such that it satisfies the properties mentioned in the beginning excluding the measurability of all sets, i.e., non-negativity, translation-invariance, σ-additivity, and the fact that $\mu([a, b]^n) = (b - a)^n$. This leads us to the so-called *Lebesgue measure.*

The σ-finiteness cannot be removed if one wants uniqueness.

Example 7. Consider the space $\mathbb{R}$. Take the algebra generated by all half-open intervals $[a, b)$ on $\mathbb{R}$. We define a pre-measure

$$\lambda(A) = \begin{cases} 0 & A = \varnothing, \\ \infty & \text{otherwise} \end{cases}$$

The Carathéodory extension gives all non-empty sets measure infinity. However, there exists another extension given by the counting measure.