Exercise Class 4

A Lebesgue Measurable Set that is not Borel

Today is all about counterexamples, so it might not surprise you that Cantor is very much involved. Much of today's class is based on this article.

1 Warm-up — Cantor's function

We start with Cantor's function (also known as Devil's staircase if you prefer to be a bit more dramatic). It is a notorious counterexample in analysis, because it challenges naive intuitions about continuity and measure.

Construction

Like the Cantor set $\mathcal{C}$ itself, there are a few different ways to think about the Cantor function. The first is to think about building it iteratively, just like one would build the Cantor set. We want to define the function on the interval $[0,1]$. We start by removing the first interval in the construction of the Cantor set and say that the function takes the value $1/2$ on said interval, i.e., $f(x) = 1/2$ on $(1/3, 2/3)$. Next, we define what it does on the next intervals we remove in the construction of C. We map the interval $(1/9, 2/9)$ to $1/4$ and $(7/9, 8/9)$ to $3/4$. We keep following this patter: every interval that is removed to construct the Cantor set gets sent to some fraction with a denominator that is a power of 2. The rest of the points just "fill in the gaps".

There is another slick way of describing the Cantor function; one that is probably a bit easier to work with. Recall, that the Cantor set allows for a simple representation if one is willing to think in base 3.

To formally define the Cantor function $c : [0,1] \to [0,1]$, let $x \in [0,1]$. We obtain $c(x)$ by the following steps:

i) Express x in base 3, i.e. $x = \sum_{n=1}^{\infty} a_n 3^{-n}$ where $a_n \in \{0,1,2\}$ for all n.

ii) If x contains a 1, truncate it at the first 1, or equivalently, replace all the following digits with 0s. If the number does not contain a 1, leave it alone for now.

iii) Replace any remaining 2s with 1s.

iv) Interpret the result in base 2 instead of base 3. The result is $c(x)$.

Example. Let us consider a few examples to make the construction more tangible.

- $1/2$ is written as $0.111\ldots$ in base 3. We truncate it at the first 1 to get 0.1 (or $0.1000\ldots$), and then we interpret that number in base 2. We get $1/2$ again, so $c(1/2) = 1/2$).
- $1/4$ in base 3 is $0.02020202\ldots$. There is no 1, so at the next step it is still $0.02020202\ldots$. This is then rewritten as $0.01010101\ldots$. When interpret this as a binary number, it corresponds to $1/3$, thus $c(1/4) = 1/3$.

Equivalently, if $\mathcal{C}$ is the Cantor set on $[0,1]$, the Cantor function $c : [0,1] \to [0,1]$ can be defined as

$$c(x) = \begin{cases} \sum_{n=1}^{\infty} a_n 2^{-(n+1)} & x = \sum_{n=1}^{\infty} a_n 3^{-n},\ a_n \in \{0,2\}, \\ \sup_{y \le x, y \in \mathcal{C}} c(y), & x \notin \mathcal{C}. \end{cases}$$

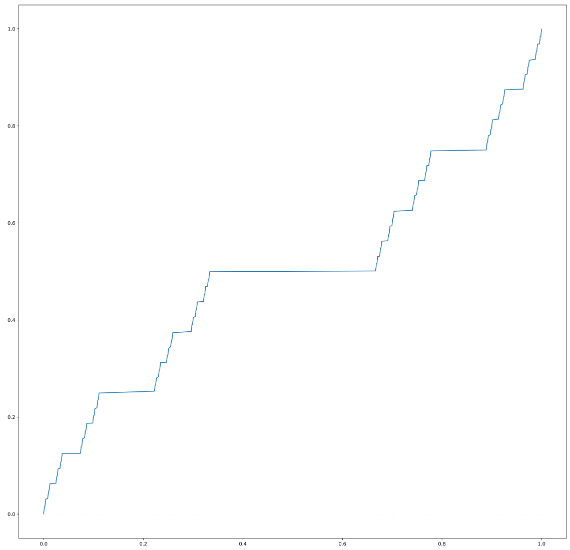

Figure 1: A plot of the Cantor fuction

Properties

The function is continuous everywhere. The continuity is not too much of a miracle. The derivative is where things get interesting.

Recall the fundamental theorem of calculus. The theorem states that for “nice enough” functions, integration and differentiation are “inverse” operations in some sense. In particular,

$$\int_a^b f'(x) = f(b) - f(a).$$

The Cantor function helps us to understand what “nice enough” means.

At every point $x \notin \mathcal{C}$ the function is constant and hence its derivative is zero. As was shown in the lecture, the Cantor set is a Lebesgue nullset, so the function has derivative 0 *almost everywhere*. For $x \in \mathcal{C}$, the function behaves completely differently. The best-behaved points in the Cantor set have an “infinite” derivative (or equivalently a vertical tangent line). The more pathological ones do not even have that.

In mathematics, “almost everywhere” is good enough for many things. However, Cantor’s function presents its limitations. The function is constant almost everywhere and yet manages to go from 0 to 1 in a continuous manner. It would be tempting to think that if a function is continuous everyhwere and differentiable almost everywhere, it should be nice enough for the fundamental theorem of calculus to be meaningful. The Cantor function shows us that those criteria are not sufficient. The derivative is zero almost everywhere, hence the integral is 0, but the starting value of c is 0 and the terminal value is 1. That is,

$$\int_0^1 c'(x)dx = 0 \neq 1 = c(1) - c(0).$$

Remark. The Cantor function can be seen as a cumulative distribution function. However, as can be seen in the previous discussion, the Cantor function cannot be represented as the integral of a probability density function. Again, this is due to the fact, that the function is not the integral of its derivative even though the derivative exists almost everywhere. One would need a function which is zero almost everywhere and yet integrates to 1, which is not possible with our notion of integration. On another note, this distribution function has no discrete part and hence the corresponding measure is atomless. This is why there are no jump discontinuities in the function. Any jump in the function would correspond to an atom in the measure. It turns out that atomless measures actually have a continuum of values. If μ is an atomless measure and A is a measurable set with $\mu(A) > 0$, then for any $b \in \mathbb{R}$ with $\mu(A) \geq b \geq 0$ there exists a measurable subset B of A such that $\mu(B) = b$. This statement has been proven for the Lebesgue measure in Exercise 3.1

Remark. A similar construction allows us to define *Volterra's function* V which is a function that is differentiable everywhere and has a derivative V' that is bounded everywhere. However, V' is not Riemann-integrable.

2 Constructing a Measurable Non-Borel set

The goal is to construct a Lebesgue measurable set which is not a Borel set. Let $\mathcal{B}$ be the Borel σ-algebra and Σ the σ-algebra of Lebesgue measurable sets. There is a neat cardinality argument to see that $\mathcal{B} \subsetneq \Sigma$.

One can show that there are only $2^{\aleph_0}$ Borel sets. But since the Cantor set is Borel (it is closed) and of measure zero, every subset of $\mathcal{C}$ is Lebesgue measurable (with measure zero). Then again, the Cantor set has cardinality $2^{\aleph_0}$, whence it has $2^{2^{\aleph_0}}$ subsets — all of which are Lebesgue measurable. Therefore, most of them are not Borel sets.

To construct such a set, we begin by defining a function $f : [0,1] \to [0,2]$ by

$$f(x) = c(x) + x,$$

where $c : [0,1] \to [0,1]$ is the Cantor function. The graph of f looks much like the Devil's staircase, except the horizontal lines are now all tilted with a slope of 1.

Claim 1: The function f is a homeomorphism.

Proof. Recall that a homoemorphism is a continuous map that has a continuous inverse function.

- f is strictly increasing, since $f' = 1$ almost everywhere;
- f is continuous, since both c and x are continuous;
- f^{-1} exists, since f is injective by monotonicity and surjective by continuity (intermediate value theorem together with $f(0) = 0$ and $f(1) = 2$);
- f^{-1} is continuous.
  Let $h = f^{-1} : [0,2] \to [0,1]$ and suppose $U \subset [0,1]$ is open. Then $[0,1] \setminus U$ is compact and hence closed and bounded. Since f is continuous, $f([0,1] \setminus U)$ is also compact and hence closed and bounded. But we can rewrite this as

  $$f([0,1] \setminus U) = f([0,1]) \setminus f(U) = [0,2] \setminus f(U) = [0,2] \setminus h^{-1}(U),$$

  which allows us to conclude that $h^{-1}(U)$ is open.

□

Claim 2: The function f maps the intervals of $[0,1]$ which are removed during the construction of $\mathcal{C}$ to intervals of $[0,2]$ of the same length.

Proof. Recall that c is constant on the intervals which are removed during the construction of $\mathcal{C}$. That is, for any interval $(a,b) \subset [0,1] \setminus \mathcal{C}$, we have $c(a) = c(b)$ and thus

$$\mu\big((f(a), f(b))\big) = f(b) - f(a) = c(b) + b - c(a) - a = b - a.$$

□

Lemma. *Every set in* Σ *with positive measure contains a non-measurable subset.*

Proof. Exercise 4.5. □

Claim 3: $\mu(f(\mathcal{C})) = 1$

Proof. Claim 2 implies that

$$\mu\big(f([0,1] \setminus \mathcal{C})\big) = \mu\big([0,1] \setminus \mathcal{C}\big) = 1.$$

Since $[0,2] = f(\mathcal{C}) \sqcup f([0,1] \setminus \mathcal{C})$, we see that

$$2 = \mu([0,2]) = \mu(f(\mathcal{C})) + \mu(f([0,1] \setminus \mathcal{C})) = \mu(f(\mathcal{C})) + 1$$

and hence

$$\mu(f(\mathcal{C})) = 1.$$

□

The Lemma and Claim 3 imply that $f(\mathcal{C})$ contains a non-measurable set N.

Claim 4: $f^{-1}(N)$ is Lebesgue measurable but not Borel.

Lemma. *A continuous and strictly increasing function defined on an interval maps Borel sets to Borel sets.*

Proof. Let f be any continuous, strictly increasing function on some interval. The argumentation above allows us to conclude that f is a homeomorphism. This enables us to show that f maps Borel sets to Borel sets. To do so, it is sufficient to prove that for any continuous function g, the set

$$\mathcal{A} = \{E : g^{-1}(E) \in \mathcal{B}\}$$

is a σ-algebra containing all open sets. Once we prove that, we can conclude that $\mathcal{A}$ contains all Borel sets and therefore, taking $g = f^{-1}$, we obtain $(f^{-1})^{-1}(E) = f(E)$ is Borel for any Borel set E.

The proof that $\mathcal{A}$ is a σ-algebra is, in fact, the same as in Exercise 2.1 (b). □

We can now prove the claim.

Proof of Claim 4. Since $N \subset f(\mathcal{C})$, we know that $f^{-1}(N) \subset \mathcal{C}$ is measurable and has measure zero as the subset of a nullset[1]. Moreover, we can show that $f^{-1}(N)$ is not a Borel set. Assume by contradiction that $f^{-1}(N) \in \mathcal{B}$. By our previous lemma, f maps this set to a Borel set. However, $f(f^{-1}(N)) = N$ is not measurable by assumption; a contradiction! □

[1]Here we use that the Lebesgue measure is complete. That is, every subset of a nullset is measurable and has measure zero.