Exercise Class 5

Fractals and the Hausdorff Dimension

The following notes try to provide heuristic methods to understand fractal sets and the Hausdorff dimension. Thus they are far from being rigorous.

In the last few weeks, we have seen that the Cantor set satisfies a peculiar set of properties. It is an uncountable set of measure zero. Moreover, as you might know from topology, it is a set that is nowhere dense and yet has no isolated points (i.e., it is a perfect set). There is an additional interesting property, namely its *self-similarity*. It is self-similar because it is equal to two copies of itself, each copy scaled by a factor of $1/3$[1]. This self-similarity makes the Cantor set a prototype of a *fractal set*. What does that exactly mean?

Understanding Fractals

There is not really an agreed upon pedantic definition for a fractal. Yet, most people have some idea what a fractal set should look like. A common misconception is that fractal sets are perfectly self-similar shapes. Mandelbrot, the father of fractal geometry, had a much broader conception in mind — a notion which is able to model the naturally occuring *roughness* of shapes such as coastlines or mountains.

In some ways, fractal geometry is the antithesis of calculus whose central assumption is that things "smooth out if you zoom in far enough" — just think of the notion of manifolds for example. For many shapes occuring in nature this assumption is overly idealized. A famous example of the limitations of that assumption is the so-called *Coastline paradox* — the counterintuitive observation that the coastline of a landmass does not have a well-defined length. The closer one looks, the longer the coastline gets. This is due to the fractal curve-like properties of coastlines.

Even though perfectly self-similar shapes make for a more restrictive notion than fractal sets, they are a good starting point to understand fractals.

Fractal Dimension

The technical definition of fractals, at least as Mandelbrot phrased it, has to do with fractal dimension, or more specifically, with Hausdorff dimension. Mandelbrot defined fractal as follows: "A fractal is by definition a set for which the Hausdorff-dimension strictly exceeds the topological dimension."[2] Dimension is a very intuitive idea in the real world, but in order to get a handle on its generalization (the Hausdorff dimension), we have to change the way we think about it. Instead of thinking about dimension in the way it was defined in linear algebra, consider it as some sort of scaling property.

More precisely, for a geometric object M and a scaling factor $s \in \mathbb{R}$, the dimension satisfies the relation

$$\mu(sM) = s^D \mu(M),$$

where μ is assumed to be some sensible notion of volume here such that $\mu(M) \neq 0, \infty$. Otherwise, the equation would be quite meaningless.

[1] Check out this gif, which makes the self-similarity of the Cantor set apparent.

[2] He later considered this definition as too restrictive and simplified it even further to "A fractal is a shape made of parts similar to the whole in some way." Still later, Mandelbrot proposed to "use fractal without a pedanctic definition and use fractal dimension as a generic term applicable to all the variants."

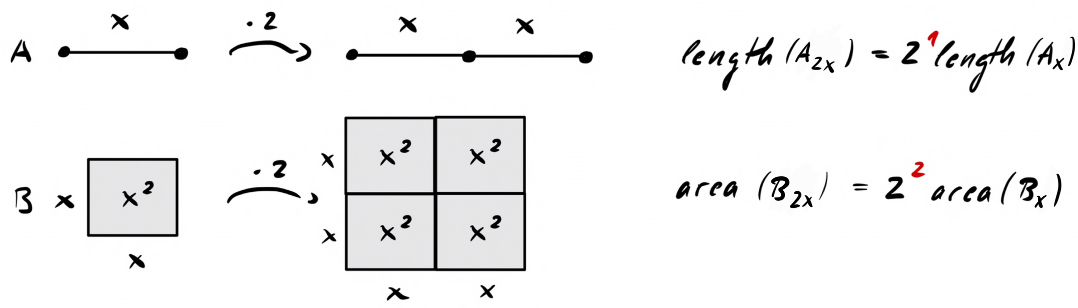

Figure 1: Scaling behaviour compared to traditional notion of dimension.

It can be easily seen that such an equation holds for the line ($D = 1$), the square ($D = 2$) and the cube ($D = 3$). See Figure 1.

Assuming we have a useful measure for the Cantor set, we can derive a similar equation as above. We have seen in the introduction that the Cantor set is equal to two copies of itself scaled by $1/3$, or equivalently

$$\mu(3 \cdot \mathcal{C}) = 2 \cdot \mu(\mathcal{C}).$$

We obtain that

$$3^D = 2 \iff D = \frac{\log(2)}{\log(3)} \approx 0.6309...$$

So in some sense, the Cantor set is neither 0-dimensional, nor 1-dimensional. That is, neither the 0-dimensional notion of volume (the counting measure) nor the 1-dimensional one (Lebesgue) captures the structure of the Cantor set. The same holds, if one thinks of the coastline example. Apparently, the length of coastlines is infinite, and yet the area is 0. Instead, what one wants, is whatever the D-dimensional analogue of length is. This will then be our measure μ.

Box-counting Dimension

In our (highly non-rigorous) computations of the fractal dimension, we relied on the self-similarity of the shapes in consideration. This is very restrictive since most shapes are not self-similar. Take the unit disk for example. We know that scaling the radius by 2 increases the area of the disk by 4 (hence the disk is a 2-dimensional object). However, it is not possible to rebuild the rescaled disk using 4 copies of the initial disk.

Just as with Carathéodory extensions, we use coverings (see Theorem 1.2.17 in the Lecture Notes). One possible choice of coverings would be to use boxes and then to simply count the number of boxes it touches (see Figure 2). With this box-counting procedure, we can proceed similarly as before. We can rescale the geometric object in question and check the number of boxes it touches after that. Leaving out some technical details, this then leads to what is known as the *box-counting dimension* or *Minkowski dimension*. The box-counting dimension thus gives us a quantitative way to describe roughness that persists on many different scales. And this is the notion of self-similarity one should have in mind when thinking of fractals.

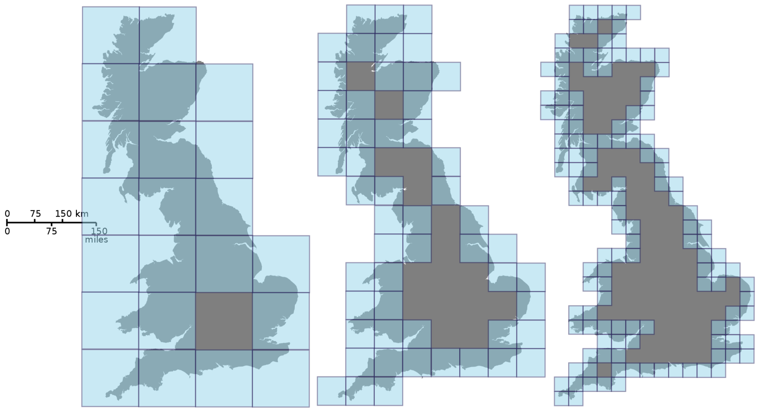

Figure 2: Box-counting dimension of the coast of Great Britain [MD10]

We have yet to define the Hausdorff dimension which Mandelbrot refers to in his definition. This will be the topic of the next section.

Hausdorff measure and dimension

The Hausdorff dimension is similar to the box-counting dimension explained before. But in some sense, it counts using balls instead of boxes. In many cases the Hausdorff dimension coincides with the box-counting dimension. However, the Hausdorff dimension is slighly more general, at the cost of being slightly more difficult to describe.

Recall that for the dimension to make sense, we needed an appropriate measure such that $\mu(A) \neq 0, \infty$ for the set A in question. We now want to construct that measure.

Definition 1. For $s \geq 0$, $\delta > 0$ and $\varnothing \neq A \subseteq \mathbb{R}^n$, we set

$$\mathcal{H}^s_\delta(A) = \inf\left\{ \sum_{k\in I} r_k^s,\ A \subseteq \bigcup_{k\in I} B(x_k, r_k),\ 0 < r_k < \delta \right\}.$$

We also set $\mathcal{H}^0_\delta(\varnothing) = 0$. The set of indices I is at most countable.

This is a non-increasing function of δ Hence its monotone limit exists.

Definition 2. We call $\mathcal{H}^s$ the *s-dimensional Hausdorff measure* on $\mathbb{R}^n$, where

$$\mathcal{H}^s(A) = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A)$$

for any $A \subseteq \mathbb{R}^n$.

Remark. Observe that $\mathcal{H}^0$ is the counting measure.

The following lemma is crucial in the definition of the Hausdorff dimension.

Lemma 3. *Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < \infty$. It holds*

i) $\mathcal{H}^s(A) < \infty \implies \mathcal{H}^t(A) = 0.$

ii) $\mathcal{H}^t(A) > 0 \implies \mathcal{H}^s(A) = \infty.$

If we consider the Hausdorff measure of a fixed set A as a function of s, this lemma tells us that there is at most one interesting value for s. Suppose that for some $s \geq 0$, we have $0 < \mathcal{H}^s(A) < \infty$, then for all $t \neq s$, the measure $\mathcal{H}^t(A)$ is either 0 or ∞. This value s is what we choose to define as dimension.

Definition 4. The *Hausdorff dimension* of a set $A \subseteq \mathbb{R}^n$ is defined to be

$$\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} \; = \sup\{s \geq 0 : \mathcal{H}^s(A) = \infty\}.$$

Note that this definition of dimension allows for non-integer dimensions. Intuitively, we try to measure the set A in every possible dimension and then pick the "best" one. This is what we then call the Hausdorff dimension.

The Hausdorff dimension is a measure of roughness. For sufficiently smooth shapes, the Hausdorff dimension is an integer which coincides with the topological dimension. Note, however, that sets with non-integer Hausdorff dimensions occur very often in nature and sets with Hausdorff dimensions equal to their topological dimension tend to be man-made.

See here for a list of fractals and their Hausdorff dimension.

Sources

[MD10] Alexis Monnerot-Dumaine. *Fractal dimension: Covering of a fractal set (here, the coast of Great Britain) by grids of decreasing sizes.* Available at `https://commons.wikimedia.org/wiki/File:Great_Britain_Box.svg`, last visited 29.03.2021. 2010.