Exercise Class 7

Almost Everywhere

The content this week is taken from [BSU96].

Last week, we have introduced the notion of *measurable functions.* Let μ be a measure on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ μ-measurable.

Definition 1. A function $f : \Omega \to [-\infty, \infty]$ is called μ-measurable if

1. $f^{-1}(\{+\infty\})$ and $f^{-1}(\{-\infty\})$ are μ-measurable.
2. f^{-1} is μ-measurable for every $U \subseteq \mathbb{R}$ open.

Remark. The set U in condition ii) can be replaced by Borel sets B or sets of the form $(-\infty, a)$ for all $a \in \mathbb{R}$.

Equivalence of functions

The framework of this section is as follows. We study real-valued functions defined on a measure space (Ω, Σ, μ).

Definition 2. We say that a property holds *μ-almost everywhere* if it holds on a set N^c, where $\mu(N) = 0$.

From now on, we consider functions $f : \Omega \to \overline{\mathbb{R}}$ that are μ-almost everywhere finite. That is, $\mu\{f(x) = \pm\infty\} = 0$.

Definition 3. Functions f and g are called *equivalent* if they coincide μ-almost everywhere, i.e., if $\mu(\{f \neq g\}) = 0$.

Theorem 4. *If μ is a complete measure, then any function equivalent to a measurable function is measurable.*

Proof. If g is measurable and $f = g$ μ-a.e. then the set $\{g < a\}$ is measurable for any $a \in \mathbb{R}$. But note that $\{f < a\}$ is also measurable since $\{f < a\}$ and $\{g < a\}$ only differ by a certain subset of the set $\{f \neq g\}$ of measure zero. However, this subset is measurable as the measure is assumed to be complete. ☐

Theorem 5. *Equivalence as in Definition 3 is an equivalence relation on the set of all measurable functions $\mathcal{M}(\Omega)$.*

Proof. Let us write f g if two functions are equivalent. Reflexivity and symmetry are obvious. It remains to prove transitivity. Let $f, g, h \in \mathcal{M}(\Omega)$, f g and g h. We set $N_1 = \{f \neq g\}$ and $N_2 = \{g \neq h\}$. We know that $\mu(N_1) = 0 = \mu(N_2)$. At the same time, we have that for any $x \in \Omega \setminus (N_1 \cup N_2)$, there holds $f(x) = g(x)$ and $g(x) = h(x)$. Thus $f(x) = h(x)$. Moreover, $\mu(N_1 \cup N_2) \leq \mu(N_1) + \mu(N_2) = 0$. We conclude that f h. ☐

Thanks to this theorem, we can consider the factor set $\mathcal{M}(\Omega)/$, i.e., we only distinguish between equivalence classes. When studying the theory of measurable functions and integration theory, we regularly ignore values of functions on sets of null measure. This means that a measurable function can be replaced by an arbitrary equivalent function. Once we established integration theory and introduce the L^p-spaces, this will be made more precise.

Sequences of Measurable Functions

We now look at a new type of convergence of sequences of measurable functions. Recall the notions of uniform and pointwise convergence.

Uniform convergence $f_n \rightrightarrows f$ means that

$$\forall \varepsilon > 0 \; \exists N \; \forall n > N \; \forall x \in \Omega : \left| f_n(x) < f(x) \right| < \varepsilon.$$

Pointwise convergence $f_n \to f$ means that

$$\forall x \in \Omega \; \forall \varepsilon > 0 \; \exists N \; \forall n > N : \left| f_n(x) - f(x) \right| < \varepsilon.$$

It is clear that uniform convergence implies pointwise convergence and that the converse is not true. We have seen in the lecture that the limit of a pointwise convergent sequence of measurable functions is measurable again[1].

In the theory of measurable functions (and especially in probability theory under the name *almost sure*), we often encounter the convergence of functional sequences μ-almost everyhwere. Convergence $f_n \to f$ μ-a.e. means that there exists a set $N \subset \Omega$ such that $\mu(N) = 0$ and

$$\forall x \in N^c : f_n(x) \to f(x).$$

It is fairly straightforward to see that μ-a.e. limits of measurable functions are measurable.

Example 6. Consider the sequence of functions $f_n(x) = x^n$ on $[0, 1]$. It is clear that

$$x^n \to \chi_1(x).$$

Since $\{1\}$ is a set of zero measure, we can write $\lim_{n\to\infty} x^n = 0$ μ-a.e., where μ is the Lebesgue measure.

Let us introduce one more type of convergence, which plays an important role in the study of measurable functions and will appear a bit later in the course.

Definition 7. A sequence of μ-a.e. finite measurable functions $(f_n)_n$ converges *in measure* μ to f, in short $f_n \xrightarrow{\mu} f$ as $n \to \infty$ if for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in \Omega : \left| f(x) - f_n(x) \right| > \varepsilon\}) = 0.$$

We will then be able to relate these notions of convergence.

Sources

[BSU96] Yuri M. Berezansky, Zinovij G. Sheftel, and Georgij F. Us. "Measurable Functions". In: *Functional Analysis: Vol. I.* Basel: Birkhäuser Basel, 1996, pp. 67–88. ISBN: 978-3-0348-9185-1. DOI: 10.1007/978-3-0348-9185-1_2.

[1] That follows by measurability of $\liminf f_n$ and $\limsup f_n$.