Exercise Class 8

Littlewood's Three Principles[1]

What does a measurable set or function look like? This is a somewhat futile question. After all, the Borel hierarchy is huge. Carathéodory's criterion of measurability gives us an intuition what a measurable set is capable of doing, namely splitting up arbitrary sets without changing its measure, however, it does not really tell us what a measurable set looks like.

The notions of measurable sets and measurable functions represent new tools, however, they are still related to older concepts. Littlewood thus introduced *three principles of real analysis* as heuristics to help understand the essentials of measure theory. This exercise class is devoted to study and exposit them.

> "The extent of knowledge [of real analysis] required is nothing like as great as is sometimes supposed. There are three principles, roughly expressible in the following terms:
>
> 1. Every set is nearly a finite sum of intervals.
> 2. Every function is nearly continuous.
> 3. Every convergent sequence is nearly uniformly convergent."
>
> - John Littlewood.

Naturally, the sets and functions referred to above are assumed to be measurable. The catch is in the word "nearly", which has to be understood appropriately in each context. In this class, we make these statements precise.

For the remainder of this class, let μ denote the Lebesgue measure on $\mathbb{R}$ and Σ the σ-algebra of μ-measurable sets.

Littlewood's First Principle

Theorem. *Every finite measurable set is nearly a finite sum of intervals. That is, if $E \in \Sigma$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$, there is a set F that is a finite union of open intervals such that $\mu(E \triangle F) < \varepsilon$.*

Proof. Let $E \in \Sigma$ and $\varepsilon > 0$. By Theorem 1.3.8. in the Lecture Notes, we know that there exists an open set $G \subset \mathbb{R}$ such that $E \subset G$ and

$$\mu(G \setminus E) < \frac{\varepsilon}{2}.$$

This shows in particular that $\mu(G) < \infty$. Thus G can be written as a countable union of disjoint open intervals $\{(a_n, b_n) \mid n \in \mathbb{N}\}$. Furthermore, by countable additivity of μ, we have

$$\mu(G) = \mu\Big(\bigcup_{n=1}^{\infty}(a_n, b_n)\Big) = \sum_{n=1}^{\infty}\mu((a_n, b_n))$$

or equivalently

$$\mu(G) = \lim_{k\to\infty}\sum_{n=1}^{k}\mu((a_n, b_n)).$$

[1]The notes this week closely follow [SS09].

The sum on the r.h.s. converges, so we may find N sufficiently large such that

$$\mu(G) - \sum_{n=1}^{N} \mu((a_n, b_n)) < \frac{\varepsilon}{2}.$$

Now, let $F = \bigcup_{n=1}^{N} (a_n, b_n)$. Then $F \subset G$ and F is open as the finite union of open intervals. We obtain that

$$\mu(F \setminus E) \leq \mu(G \setminus E) < \frac{\varepsilon}{2}.$$

Similarly, since $E \subset G$, we know that $E \setminus F \subset G \setminus F$ and so it follows (by using measurability of F) that

$$\mu(E \setminus F) \leq \mu(G \setminus F) = \mu(G) - \mu(F) = \mu(G) - \sum_{n=1}^{N} \mu((a_n, b_n)) < \frac{\varepsilon}{2}.$$

Since $E \setminus F$ and $F \setminus E$ are disjoint, it follows from the disjoint additivity of μ that

$$\mu(E \triangle F) = \mu\big((E \setminus F) \cup (F \setminus E)\big) = \mu(E \setminus F) + \mu(F \setminus E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

The proof can easily be generalized for $\mathbb{R}^d$. One simply has to write open sets in $\mathbb{R}^d$ as the union of disjoint (dyadic) cubes and then the proof can essentially be copied from above.

The latter principles are more striking but Littlewood's first principle is of help when thinking about measurable sets. Additionally, it provides a straightforward proof of the Riemann–Lebesgue lemma which states that the Fourier transform of an integrable function vainshes at infinity.

Lemma (Riemann–Lebesgue). *If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable, then*

$$\lim_{s \to \pm\infty} \int_{\mathbb{R}} f(x) e^{ikx} dx = 0.$$

Proof. First, we prove the result for $f = \chi_I$, where I is a bounded interval, say $I = (a, b)$. This is trivial, since

$$\int_a^b e^{ikx} dx = \frac{e^{ika} - e^{ikb}}{ik} \to 0 \quad \text{for } s \to \pm\infty.$$

Just as trivially, the result extends to any finite union of bounded intervals.

Next, consider $f = \chi_E$, where E is any Lebesgue measurable set with finite measure. Let $\varepsilon > 0$, then by Littlewood's first principle, there exists a finite union F of bounded intervals such that $\mu(E \triangle F) < \varepsilon$. By the first part of the proof, we have that

$$\left| \int_F e^{ikx} dx \right| < \varepsilon$$

for $|k|$ large enough, in which case it follows that

$$\left| \int_E e^{ikx} dx \right| \leq \left| \int_F e^{ikx} dx \right| + \left| \int_{E \triangle F} e^{ikx} dx \right| < 2\varepsilon.$$

The result thus naturally extends to any simple function.

Finally, if f is integrable, then given $\varepsilon > 0$, there exists a simple integrable function g so that

$$\int_{\mathbb{R}} \big|f(x) - g(x)\big|\, dx < \varepsilon.$$

By what we have proved before, it holds that

$$\left| \int_{\mathbb{R}} g(x) e^{ikx} dx \right| < \varepsilon$$

when $|k|$ is sufficiently large. This allows us to conclude as

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x) e^{ikx} dx \right| &\leq \left| \int_{\mathbb{R}} (f(x) - g(x)) e^{ikx} dx \right| + \left| \int_{\mathbb{R}} g(x) e^{ikx} dx \right| \\ &< \int_{\mathbb{R}} \big|f(x) - g(x)\big|\, dx + \varepsilon < 2\varepsilon. \end{aligned}$$

□

Littlewood's Third Principle

Littlewood's Third Principle states that every convergent sequence is nearly uniformly convergent. This is better known as *Egorov's theorem.*

Theorem (Egorov)**.** *Suppose $(f_k)_k$ is a sequence of measurable functions defined on a measurable set Ω with $\mu(\Omega) < \infty$, and assume that $f_k \to f$ μ-a.e. on Ω. Given $\varepsilon > 0$, we can find a compact set $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and $f_k \to f$ uniformly on K.*

Proof. See Theorem 2.3.1 in the Lecture Notes □

The reason for introducing the Third Principle before the Second one will become apparent in the next section.

Littlewood's Second Principle

The result that "every function is nearly continuous" is better known as *Lusin's theorem.*

Theorem (Lusin)**.** *Suppose f is measurable and finite valued on Ω with $\mu(\Omega) < \infty$. Then for every $\varepsilon > 0$ there exists a closed set F with*

$$F \subset \Omega \quad \textit{and} \quad \mu(\Omega \setminus F) < \varepsilon$$

and such that $f|_F$ is continuous.

By $f|_F$ we mean the restriction of f to the set F. The theorem states that if f is viewed as a function defined *only* on F, then it is continuous. However, the theorem does not make the stronger assertion that the function f defined on Ω is continuous at the points of F.

The idea is to approximate f pointwise almost everywhere with a sequence of step functions. Outside a set of small measure, the set functions are continuous. So then by Egorov's theorem, outside a set of small measure, f is the uniform limit of continuous functions and so f is continuous.

Proof. Let f_n be a sequence of simple functions so that $f_n \to f$ μ-a.e. Then we can find sets E_n so that $\mu(E_n) < 2^{-n}$ and f_n is continuous outside of E_n. By Egorov's theorem, we can find a set K on which $f_n \to f$ uniformly and $\mu(\Omega \setminus K) < \varepsilon/3$. Then we consider

$$F' = K \setminus \bigcup_{n \geq N} E_n$$

for N so large that $\sum_{n \geq N} 2^{-n} < \varepsilon/3$. Now for every $n \geq N$ the function f_n is continuous on F'. Thus f being the uniform limit of $(f_n)_n$ is also continuous on F'. To finish the proof, we merely need to approximate F' by a closed set $F \subset F$ such that $\mu(F' \setminus F) < \varepsilon/3$, which can be easily done with Littlewood's First Principle. It then follows that $\mu(\Omega \setminus F) < \varepsilon$ and $f|_F$ is continuous. $\square$

Remark. Observe that we stated Egorov's and Lusin's theorem in less generality than in the lecture. For simplicity, we assumed that f is finite everywhere.

Sources

[SS09] Elias M. Stein and Rami Shakarchi. *Real Analysis*. Princeton University Press, 2009. DOI: doi:10.1515/9781400835560.