Exercise Class 12
L^p-spaces[1]

Definition

Let $\Omega \subset \mathbb{R}^n$ and μ be a Radon measure; notions such as "measurable", "almost everywhere", etc. will always be with respect to the measure μ. Moreover, unless otherwise specified, all subsets and functions mentioned are supposed to be measurable.

We already have the notion of a summable function, which is a function $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f| \, d\mu$ is finite. More generally, given any exponent $1 \leq p < \infty$, we can define a p^{th}-power integrable function to be a function $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f|^p \, d\mu$ is finite. In short, we denote the collection of such functions by $\mathcal{L}^p(\Omega, \mu)$.

Following the "Lebesgue philosophy" that we can neglect what happens on nullsets, we declare two functions to be equivalent if they agree almost everywhere, i.e.

$$f \sim g :\iff f = g \ \mu\text{-a.e.}$$

This is, in fact, an equivalence relation (see Exercise Class 7). This allows us to define the spaces $L^p(\Omega, \mu)$ to be the space of p^{th}-power summable functions, quotiented by this equivalence relation. So, strictly speaking, an element of $L^p(\Omega, \mu)$ is not a function f, but rather an equivalence class of functions $[f|]$ which agree almost everywhere. We shall consistently abuse notation and just write f for an element of $L^p(\Omega, \mu)$ and call it a function nonetheless. For the purpose of integration, this equivalence is quite harmless, however, this means that we cannot evaluate functions at a single point x if that point has measure zero. One possible way to think about elements of L^p is that they are functions which are "unreliable" on an unknown set of measure zero.

Remark. Depending on which part of the measure space (Ω, μ) one wishes to emphasize, the space $L^p(\Omega, \mu)$ is often denoted by $L^p(\Omega)$ or $L^p(\mu)$, or even just L^p. Since the measure space (Ω, μ) is fixed in our case, we shall use the L^p abbreviation from now on.

At the moment, L^p is just a set. We can endow it with a vector space structure. The corresponding operations in L^p are defined via representatives; so $f + g$ for instance is in L^p the equivalence class associated to the sum of one representative each from the respective equivalence classes of f and g. One can define scalar multiplication in a similar way. It is tedious but straightforward to check that this makes everything well-defined.

Next, let us set up the norm structure. If $f \in L^p$, we define the L^p norm $\|\cdot\|_{L^p}$ of f to be the number

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p \, d\mu \right)^{1/p}.$$

This is a finite non-negative number by definition of L^p; in particular, we have

$$\|f^r\|_{L^p} = \|f\|_{L^{pr}}^r$$

for all $1 \leq p, r < \infty$.

We can then prove that $\|\cdot\|_{L^p}$ does indeed define a norm on the space L^p. The only non-trivial thing to prove is, in fact, the triangle inequality. Note that the L^p norms would have been seminorms, if we did not equate functions that agreed almost everywhere because in that case, $\|\cdot\|_{L^p}$ does not satisfy positive definiteness.

[1] Today's notes are partly based on [Tao11]

The L^∞ space

We can now define L^p norms and spaces in the limit $p = \infty$. We say that a function $f : \Omega \to \mathbb{R}$ is *essentially bounded* if there exists a constant C such that $\left|f(x)\right| \leq C$ for almost every $x \in \Omega$. Hence, we define

$$\|f\|_{L^\infty} := \inf\{C > 0 : |f| \leq C \ \mu\text{–a.e.}\}.$$

We then let L^∞ denote the space of essentially bounded functions, quotiented out by equivalence, and given the norm $\|\cdot\|_{L^\infty}$. It is fairly straightforward to see that this is also a normed vector space.

Let us explain, why we call this the L^∞ norm.

Example 1. Let f be given by $f = a\mathbb{1}_E$ for some constant $a > 0$ and some set E with positive finite measure. Then

$$\|f\|_{L^p} = a\mu(E)^{1/p}$$

for all $1 \leq p < \infty$ and $\|f\|_{L^\infty} = a$. Thus is this case, at least, the L^∞ norm is the limit of the L^p norms.

With suitable assumptions, one can generalize this statement.

Proposition 1: L^∞ norm

Suppose that $\mu(\Omega) < \infty$ and $f \in \cap_{p\in\mathbb{N}} L^p$ with $\sup_p \|f\|_{L^p} < \infty$, we have that $\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$.

Proof. See Exercise 12.4. □

Completeness

Once one has a vector space with a norm structure, we immediately get a metric structure, which in turn generates a topological structure in the usual manner. In particular, we say that a sequence of functions $f_n \in L^p$ converges to a limit $f \in L^p$ if $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$. We refer to this type of convergence as *convergence in* L^p (or, especially in functional analysis, *strong convergence*).

To prove statements about the L^p spaces (e.g. to show that $\|\cdot\|_{L^p}$ is a norm), one heavily relies on a few inequalities.

Proposition 2: Important Inequalities

- **(Hölder inequality)**
  Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, for all measurable real-valued functions f and g,
  $$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$
- **(Minkowski inequality)**
  Let $p \in [1, \infty]$ and $f, g \in L^p$. Then $f + g \in L^p$ and
  $$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Remark. Usually, the main idea to prove Hölder's inequality is to use Young's inequality for products. One can then use Hölder's inequality to prove the Minkowski inequality, which is the triangle inequality for L^p, $1 \leq p \leq \infty$.

The proofs of Hölder's and Minkowski's inequality both ultimately relied on convexity of various real-valued functions. One way to emphasize this, is to deduce both inequalities from Jensen's inequality.

Proof of Hölder using Jensen. Recall Jensen's inequality. Let (Ω, ν) be a probability space, i.e. $\nu(\Omega) = 1$. If h is a real valued function that is ν-summable, and if ϕ is a convex function on the real line, then:

$$\phi\left(\int_\Omega h d\mu\right) \leq \int_\Omega \phi \circ h d\mu.$$

In the probability setting, this is concisely states as $\phi(E[X]) \leq E[\phi(X)]$, where X is a ν-summable random variable. In particular, this holds for $\phi(x) = x^p$ for $p \geq 1$.

Let us now prove Hölder's inequality. Let μ be any measure, and ν be the distribution whose density w.r.t. μ is proportional to g^q, i.e.

$$d\nu = \frac{g^q}{\int g^q d\mu} d\mu.$$

Letting $h = fg^{1-q}$, we obtain

$$\begin{aligned}
\int fg d\mu &= \left(\int g^q d\mu\right) \int \underbrace{fg^{1-q}}_{h} \underbrace{\frac{g^q}{\int g^q d\mu} d\mu}_{d\nu} \\
&\leq \left(\int g^q d\mu\right) \left(\int h^p d\nu\right)^{1/p} \\
&= \left(\int g^q d\mu\right) \left(\int f^q g^{p(1-q)} \frac{g^q}{\int g^q d\mu} d\mu\right)^{1/p} \\
&= \left(\int g^q d\mu\right) \left(\int \frac{f^p}{\int g^q d\mu} d\mu\right)^{1/p} \\
&= \left(\int g^q d\mu\right)^{1/q} \left(\int f^p d\mu\right)^{1/p}.
\end{aligned}$$

To go from the third to the fourth line, we used that $\frac{1}{p} + \frac{1}{q} = 1$ and hence $p(1-q)+q = 0$. In the last line, we used $\frac{1}{p} + \frac{1}{q} = 1$ once more. □

An important corollary of Hölder's inequality is that for $p = q = 2$, we obtain the *Cauchy–Schwarz inequality*

$$\left|\int_\Omega fg d\mu\right| \leq \|f\|_{L^2} \|g\|_{L^2}.$$

The main result of the section on L^p spaces is the following.

Theorem 3: Fischer–Riesz

The spaces L^p, $1 \leq p \leq \infty$, are Banach spaces.

The previous remark on the Cauchy–Schwarz inequality actually hints at the fact that the space L^2 is even a Hilbert space, i.e. it is a vector space equipped with an inner product that is also complete (with respect to the metric induced by the inner product). The inner product in this case is given by

$$\langle f, g\rangle_{L^2} = \int_\Omega fg d\mu.$$

Linear functionals

Given an exponent $1 \leq p \leq \infty$, define the conjugate q by the formula $\frac{1}{p} + \frac{1}{q} = 1$. From Hölder's inequality, we see that for any $g \in L^q$, the function $\lambda_g : L^p \to \mathbb{R}$ defined by

$$\lambda_g(f) := \int_\Omega fg d\mu$$

is well defined on L^p; the functional is also clearly linear. Furthermore, Hölder's inequality also tells us that this functional is continuous. This stems from the fact that linear operators are continuous if they are bounded. Hence, every fuction $g \in L^q$ induces a linear functional in $(L^p)^\star$ by λ_g. A deep fact about L^p spaces is that, in most cases, the converse is true as well. That is, every element of $(L^p)^\star$ is of the form λ_g for some $g \in L^q$.

Theorem 4: Dual of L^p

Let $1 \leq p < \infty$ and let q be its conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Assume that μ is σ-finite. Then every $\lambda \in (L^p)^\star$ has the form $\lambda = \lambda_g$ for some $g \in L^q$, i.e. we have $(L^p)^\star = \{\lambda_g : g \in L^q\}$. In that sense, one can identify the dual space of L^p with L^q.

This result should be compared with the Radon–Nikodym Theorem from Exercise Class 10. Both theorems start with an abstract function (a measure $\mu : \mathcal{P}(\Omega) \to \mathbb{R}$ and a linear functional $\lambda : L^p \to \mathbb{R}$ respectively), and create a function out of it with which it can be identified. One can indeed show, that both theorems are essentially equivalent.

Sources

[Tao11] Terence Tao. *An Epsilon of Room, I: Real Analysis*. American Mathematical Society, 2011.