Exercise Class 13

The Fundamental Theorem of Calculus for Lebesgue Integration

Introduction

We recall the Fundamental Theorem of Calculus.

Theorem 1: Fundamental Theorem of Calculus

Part I of the theorem says that if f is continuous on $[a, b]$, then the function defined by

$$F(x) = \int_a^x f(t)dt$$

is differentiable, and $F'(x) = f(x)$ for all $x \in [a, b]$. In particular, $F \in C^1([a, b])$. Part II of the Theorem states that given any $F \in C^1([a, b])$, we have that

$$\int_a^b F'(x)dx = F(b) - F(a).$$

We now explore the question of the extent to which this theorem continues to hold when the differentiability or itengrability conditions on F, F', or f are relaxed.. To generalize the first part, we may ask ourselves the follwing question:

- Suppose f is integrable on $[a, b]$ and F is its indefinite integral $F(x) = \int_a^x f(t)dt$. Does this imply that F is differentiable and that $F' = f$?

The second part of the theorem is usually formulated in a slightly more general way. Namely, if a real function F on $[a, b]$ admits a derivative $f(x)$ at every point $x \in [a, b]$ and if this derivative f is (Riemann-)integrable, then

$$F(b) - F(a) = \int_a^b f(t)dt.$$

The question arises as to when F admits such a derivative. Clearly, $F \in C^1$ is a sufficient condition, however, it is not necessary. What happens if F is simply differentiable, or just differentiable a.e. with a Lebesgue integrable derivative f? Hence we ask ourselves

- What conditions on a function F on $[a, b]$ guarantee that $F'(x)$ exists (for a.e. x), that this function is integrable, and that moreover

$$F(b) - F(a) = \int_a^b F'(x)dx\ ?$$

The First Fundamental Theorem

We briefly discuss the first problem. If f is given on $[a, b]$ and integrable on that interval, we let

$$F(x) = \int_a^x f(t)dt.$$

To deal with $F'(x)$, we recall the definition of the derivative as

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

We note that this quotient takes the form

$$\frac{1}{h} \int_x^{x+h} f(t)dt = \frac{1}{\mathcal{L}(I)} \int_I f(t)dt,$$

where we write I for the interval $(x, x+h)$ and $\mathcal{L}$ for the Lebesgue measure.

We observe that this expression is the "average" value of f over I, and that in the limit $\mathcal{L}(I) \to 0$ (i.e. $h \to 0$), we might expect that these averages tend to $f(x)$. Reformulating the question slightly (and in higher dimensions), we may ask whether

$$\lim_{\substack{\mathcal{L}(B) \to 0 \\ x \in B}} \frac{1}{\mathcal{L}(B)} \int_B f(t)dt = f(x), \quad \text{for a.e. } x?$$

The limit is taken as the volume of open balls B containing x goes to 0.

By these observations, it becomes apparent that the answer to our original question is closely related to Lebesgue's differentiation theorem.

Theorem 2: Lebesgue Differentiation Theorem

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then for μ-a.e. $x \in \mathbb{R}^n$ it holds

$$\lim_{r \to 0} \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} f d\mathcal{L} = f(x).$$

The Vitali covering lemma is vital to the proof of this theorem; its role lies in proving the estimate for the Hardy–Littlewood maximal function.

The Second Fundamental Theorem

We now take up the second question. We already know that differentiability a.e. with Lebesgue integrable derivative is not enough. The Cantor function[1] C on $[0,1]$ is continuous, non-decreasing, differentiable with derivative 0 almost everyhwere, however, $C(0) = 0$ and $C(1) = 1$ and thus

$$\int_0^1 C'(x)dx = 0 \neq 1 = C(1) - C(0).$$

In view of this counterexample, we see that we need to add additional hypothesis to the fuction F before we can recover the second fundamental theorem.

Definition 3: Absolute Continuity

A function F is called *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite collection of disjoint intervals (a_k, b_k), $k = 1, 2, \ldots, n$, we have

$$\sum_{k=1}^n (b_k - a_k) < \delta \implies \sum_{k=1}^n \left|F(b_k) - F(a_k)\right| < \varepsilon.$$

[1]See Exercise Class 4

For a compact interval, we have that

$$F \text{ Lipschitz cont.} \implies F \text{ absolutely cont.} \implies F \text{ uniformly cont..}$$

For absolutely continuous functions, we can recover the second fundamental theorem of calculus.

Theorem 4: Second Fundamental Theorem

Suppose F is absolutely continuous on $[a, b]$. Then F' exists almost everywhere and is summable. Moreover,

$$F(x) - F(a) = \int_a^x F'(t)dt, \quad \forall a \leq x \leq b.$$

Conversely, if f is summable on $[a, b]$, then there exists an absolutely continuous function F such that $F'(x) = f(x)$ almost everywhere, and in fact, we may take $F(x) = \int_a^x f(t)dt$.

This theorem is extremely important in Lebesgue integration theory and there are several ways of proving it. One of the better known proofs relies on the non-trivial Vitali Covering Lemma. One particular approach can be seen in [Rud87] which treats the subject by differentiating measures and thus makes use of the Radon–Nikodym theorem (somewhat similar to the approach in the lecture).

We will briefly hint at how the Radon–Nikodym theorem implies the Fundamental Theorem. Recall that we first defined the notion of absolute continuity for measures[2]. Absolute continuity for functions is closely related to that.

Proposition 5: Relation between the two notions of absolute continuity

Let Λ_F denote the Lebesgue–Stieltjes measure corresponding to F. Then the following are equivalent:

1. Λ_F is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$.
2. F is absolutely continuous.

Now recall that if $\Lambda_F \ll \mathcal{L}$, the Radon–Nikodym theorem gives us a function $f \in L^1$ — the so-called Radon–Nikodym derivative — such that

$$\Lambda_F(A) = \int_A f d\mathcal{L}.$$

If A is an interval (a, b) in $\mathbb{R}$, then $\Lambda_F(A)$ evaluates to $F(b) - F(a)$. So the Radon–Nikodym theorem yields the second fundamental theorem of calculus, and the Radon–Nikodym derivative turns out to be the classical derivative[3].

Note moreover, that we are being non-rigorous here. Most notably, we disregard the fact that we only defined the Lebesgue–Stieltjes measure for non-decreasing functions in this lecture. Nonetheless, we stated the results in a slightly more general fashion.

[2]See Exercise Class 10.

[3]Of course, it isn't pure coincidence that this function is called derivative.

References

[Rud87] W. Rudin. *Real and Complex Analysis*. Mathematics series. McGraw-Hill, 1987. ISBN: 9780071002769. URL: https://books.google.ch/books?id=NmW7QgAACAAJ.