Mathematics of Machine Learning
Homework 2

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Try to solve the questions before looking to the answers. Every item must be proved rigorously.

Problem 1

Prove Proposition 3.1 in the lecture notes.

Problem 2

Let $A \in \mathbb{R}^{n \times n}$ be a matrix with non-zero singular values ordered as usual $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$.

(a) Prove that A is invertible.

(b) Compute the condition number of the matrix A, namely $\|A\|\|A^{-1}\|$, in terms of singular values of A.

(c) Conclude that the conditional number of A is one, if and only if A is a scaled isometry.

As always, $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n | \, \|x\|_2 = 1\}$ is the standard Euclidean sphere and $\|A\| := \sup_{x \in \mathbb{S}^{n-1}} \|Ax\|_2$ is the operator norm of A.

Problem 3

Let $A, B \in \mathbb{R}^{m \times n}$ be two arbitrary matrices. Find the solution, in terms of A and B, of the following optimization problem:

$$\arg \min_{\Omega \in \mathcal{O}(m)} \quad \|\Omega A - B\|_F$$

Here $\mathcal{O}(m)$ denotes the set of all $m \times m$ orthogonal matrices and $\|.\|_F$ the usual Frobenius norm.
Hint: Recall that the Frobenius norm is induced by the trace inner product in the appropriate space of matrices.