Mathematics of Machine Learning
Homework 4

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Try to solve the questions before looking to the answers. Every item must be proved rigorously. Starred problems are harder.

Problem 1

Let G be a graph. Prove that the dimension of the nullspace of the Laplacian matrix of G counts the number of connected components of G.

Problem 2

Let G be a graph with Laplacian matrix $L(G) \in \mathbb{R}^{n \times n}$ whose eigenvalues are $\{0, \lambda_2, \dots, \lambda_n\}$. The complement graph associated with graph G, namely G^c, is defined as the graph with same vertices as G in which two vertices are connected if and only they are not connected by an edge in G.

(a) Prove that the eigenvalues of $L(G^c)$ are $\{0, n - \lambda_n, \dots, n - \lambda_2\}$

(b) If n is an eigenvalue of $L(G)$, then G^c is disconnected.

(c) If n is an eigenvalue of $L(G)$, then the number of zero eigenvalues of $L(G)$ is exactly one.

Problem 3

Prove that a collection of vectors $\{\phi_1, \dots, \phi_m\}$ in $\mathbb{C}^d$ is a frame if and only if its spans the entire space.