Mathematics of Machine Learning
Homework 6

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Try to solve the questions before looking to the answers. Every item must be proved rigorously. Starred problems are harder.

Problem 1

In this exercise we will only focus on $\mathbb{R}^2$. A set $B \subset \mathbb{R}^2$ is said to be convex if for all $x, y \in B$ and $\lambda \in [0,1]$, the convex combination $\lambda x + (1-\lambda)y$ belongs to B. Denote $\|x\|_p := (|x_1|^p + |x_2|^p)^{1/p}$ for $p > 0$ and $\|x\|_0$ denotes the number of nonzero entries of the vector.

(a) For which values of $p \geq 0$ is the unit ball of $\|.\|_p$, i.e $B_p := \{x \in \mathbb{R}^2 | \ \|x\|_p \leq 1\}$, convex?

(b) Give a justification for the term "convex relaxation" when we minimize $\|x\|_1$ instead of $\|x\|_0$.

Problem 2

Let $A \in \mathbb{C}^{d\times N}$ be a matrix. Suppose that every s-sparse vector x can be uniquely recovered by A via $\|.\|_0$ minimization.

(a) Prove that every $2s$ columns of A are linearly independent.

(b) Prove that $d \geq 2s$.

(c) Prove that if a matrix $B \in \mathbb{C}^{d\times N}$ satisfies the condition that every $2s$ columns are linearly independent, then every s-sparse vector x can be uniquely recovered by B via $\|.\|_0$ minimization.

Problem 3

* For a set S, we denote by $\overline{S}$ its complement and by $|S|$ its cardinality. In the lecture, we have seen conditions that guarantee recovery of sparse vectors via ℓ_1 minimization. The goal of this exercise is to prove one direction of a similar condition for the recovery of nonnegative signals.
Let $s < d < N$. Given $A \in \mathbb{R}^{d\times N}$, prove that every nonnegative s-sparse vector $x \in \mathbb{R}^N$ is the unique solution of

$$\text{minimize}_{z\in\mathbb{R}^N} \|z\|_1 \text{ subject to } Az = Ax \text{ and } z \geq 0,$$

if the following property holds:

$$v_{\overline{S}} \geq 0 \Rightarrow \sum_{j=1}^{N} v_j > 0,$$

for all $v \in \ker(A)/\{0\}$ and $S \subset [N]$ with $|S| \leq s$.