Mathematics of Machine Learning
Homework 9

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Try to solve the questions before looking to the answers. Every item must be proved rigorously. Starred problems are harder.

Problem 1

Let $X = \mathbb{R}^2$ and $Y = \{0, 1\}$. Let $\mathcal{F}$ be the set of classifiers corresponding to all concentric circles in the plane centered at the origin, precisely

$$f_r(x) := \begin{cases} 1, \|x\| \leq r \\ 0, \text{otherwise} \end{cases}$$

Prove that $\mathcal{F}$ is PAC-learnable and give an upper bound to the sample complexity.

Problem 2

Assume the data is linearly separable with the margin γ. Let $\hat{f}_S$ be the classifier returned by the Perceptron algorithm after training over the sample S (drawn from some unknown distribution) with size n and running through it until the algorithm makes a pass over the sample with no mistakes. Give a bound for the expected risk via the Leave-One-Out argument.

Problem 3

Let $F(t) := \mathbb{P}(X \leq t)$ be the cumulative distribution function of a random variable X and let $\hat{F}_n$ be the empirical cumulative distribution function with respect to an i.i.d sample $X_1, \dots, X_n$, i.e, consider i.i.d random variables $X_1, \dots, X_n$ with the same distribution of X, the empirical cumlative distribution with respect to such sample is given by

$$\hat{F}_n(t) := \frac{\sum_{i=1}^{n} \mathbb{1}_{X_i \leq t}}{n}.$$

(a) Prove that $\hat{F}_n$ converges uniformly to F in probability. That is,

$$\sup_{t \in \mathbf{R}} |\hat{F}_n(t) - F(t)| \xrightarrow{P} 0.$$

(Hint: Apply the uniform law of large numbers together with the bound for the shatter function of intervals)

(b) * Prove that $\hat{F}_n$ converges uniformly to F almost surely. (Hint: Use Borell-Cantelli lemma)