Mathematics of Machine Learning
Homework 4 - Solutions

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Try to solve the questions before looking to the answers. Every item must be proved rigorously. Starred problems are harder.

Problem 1

Let G be a graph. Prove that the dimension of the nullspace of the Laplacian matrix of G counts the number of connected components of G.

Solution 1

Suppose that G has K connected components. Without loss of generality, assume that the vertices are ordered according to the number of connected components they belong to. In such case, the Laplacian matrix of G has a block diagonal form with blocks $L_1, \dots, L_K$ where each block i is the Laplacian matrix corresponding the connected component i. Since each component is connected, every block has exactly one zero eigenvalue because the second smallest eigenvalue is non-zero (Theorem 6.6 from lecture notes). As in all block diagonal matrices, the eigenvalues of the entire matrix is the union of the eigenvalues of the block diagonal matrices. It follows that the nullspace of $L(G)$ is of dimension K because it contains exactly K zero eigenvalues.

Problem 2

Let G be a graph with Laplacian matrix $L(G) \in \mathbb{R}^{n\times n}$ whose eigenvalues are $\{0, \lambda_2, \dots, \lambda_n\}$. The complement graph associated with graph G, namely G^c, is defined as the graph with same vertices as G in which two vertices are connected if and only they are not connected by an edge in G.

(a) Prove that the eigenvalues of $L(G^c)$ are $\{0, n - \lambda_n, \dots, n - \lambda_2\}$

(b) If n is an eigenvalue of $L(G)$, then G^c is disconnected.

(c) If n is an eigenvalue of $L(G)$, then the number of zero eigenvalues of $L(G)$ is exactly one.

Solution 2

(a) Clearly 0 is an eigenvalue of $L(G^c)$. For $i \geq 2$, let v_i be the eigenvector corresponding to the eigenvalue λ_i and observe that all v_i are orthogonal to $(1, \dots, 1)$. Moreover, by definition of complement graph we have $L(G) + L(G^c) = nI_n - J$ where J is all ones matrix. Then we write

$$L(G^c)v_i = (nI_n - J - L(G))v_i = nv_i - \lambda_i v_i = (n - \lambda_i)v_i.$$

It is easy to see that v_i is also an eigenvector of $L(G^c)$ associated with the eigenvalue $(n - \lambda_i)$.

(b) If n is an eigenvalue of the Laplacian of G, then the second smallest eigenvalue of $L(G^c)$ is zero by letter "a". It follows from Theorem 6.6 of the lecture notes that G^c is disconnected.

(c) If n is an eigenvalue of the Laplacian of G, then G^c is disconnected by letter "b". The graph induced by the union of G and G^c is the complete graph with n vertices (by definition of complement graph), we claim that G must be connected. The proof follows from the claim by applying Theorem 6.6 from the lecture notes to conclude that $\lambda_2(G) > 0$. It remains to prove the claim. For two arbitrary vertices x and y in G we have to prove that there exits a path between them. If x and y lie on two different connected components of G^c, then exists an edge e that connects x to y (length one path). On the other hand, if x and y lie on the same connected component of G^c, choose z an arbitrary vertex of G^c that lies on a different connected component (such component must exist because G^c is not connected). By the same reason as before there exist two edges, namely e_1 that connects x to z and e_2 that connects y to z. Therefore the path $\{e_1, e_2\}$ connects x to y. Since x and y are arbitrary vertices in G, we conclude that G is a connected graph.

(Observe that in general, the union of two disconnected graphs does not necessarily need to be connected)

Problem 3

Prove that a collection of vectors $\{\phi_1, \dots, \phi_m\}$ in $\mathbb{C}^d$ is a frame if and only if its spans the entire space.

Solution 3

If the collection is a frame but it does not span the entire space, then there exist an non-zero vector $a \in \mathbb{C}^d$ that is orthogonal to all ϕ_i. By definition of frame, we get that

$$A\|a\|_2^2 \leq \sum_{i=1}^{m} |\langle \phi_i, a \rangle|^2 = 0.$$

This implies that $A = 0$, it contradicts the fact that $\{\phi_1, \dots, \phi_m\}$ is a frame. Now assume that $\{\phi_1, \dots, \phi_m\}$ spans the entire space. We may assume that all ϕ_i are non-zero. Clearly, $B := \sum_{i=1}^{m} \|\phi_i\|_2^2$ is an upper bound. Consider the continuous map Ψ that maps $x \in \mathbb{C}^d$ into $\Psi(x) = \sum_{i=1}^{m} |\langle \phi_i, x \rangle|^2$. Since the unit ball B_1 in $\mathbb{C}^d$ is compact because the space is finite dimensional, by the continuity of the map, there exists an unit vector y that satisfies the infimum below

$$A := \sum_{i=1}^{m} |\langle y, \phi_i \rangle|^2 = \inf_{x \in B_1} \Psi(x).$$

If $A = 0$, then y is orthogonal to all ϕ_i contradicting the fact that the collection $\{\phi_1, \dots, \phi_m\}$ spans the entire space. So $A > 0$ and therefore $\{\phi_1, \dots, \phi_m\}$ is indeed a frame with frame bounds A and B.