Mathematics of Machine Learning
Homework 5 - Solutions

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Try to solve the questions before looking to the answers. Every item must be proved rigorously. Starred problems are harder.

Problem 1

Given m and d the smallest worst case coherence is the minimum worst case coherence among all frames of m unit norm vectors in d dimensions. Can you find the smallest worst case coherence for the following cases:

(a) 2 vectors in 2 dimensions ($d = 2$, $m = 2$)

(b) 3 vectors in 2 dimensions ($d = 2$, $m = 3$)

Solution 1

(a) By the definition of coherence, it is a nonnegative quantity. We have two vectors in a two dimensional space, so we can take an orthonormal basis and it gives zero coherence.

(b) Let's consider the real case, the complex case is analogous. The intuition for this problem goes as follows: Draw three vectors in the plane $\mathbb{R}^2$ and consider the angles between such vectors. We want that the absolute value of the cosine of such angles to be minimized, it is intuitive that it should be the case in which all angles are equal. We proceed to prove it formally. Let $\Phi \in \mathbb{R}^{2\times 3}$ be an arbitrary matrix whose columns are ϕ_1, ϕ_2, ϕ_3. The associated frame operator is $\mathrm{S} = \Phi\Phi^*$ and consider the Gram matrix $G := \Phi^*\Phi$. The trace of S can be bounded as follows

$$3 = \mathrm{Tr}(G) = \mathrm{Tr}(\mathrm{S}) = \langle \mathrm{S}, I_2 \rangle_F \leq \|\mathrm{S}\|_F \|I_2\|_F = \sqrt{\mathrm{Tr}(SS^*)}\sqrt{2}$$

By the cyclic property of the trace,

$$\mathrm{Tr}(SS^*) = \mathrm{Tr}(GG^*) = \sum_{i,j=1}^{3} \langle \phi_i, \phi_j \rangle^2 = 3 + \sum_{i\neq j}^{3} \langle \phi_i, \phi_j \rangle^2$$

The summation in the right hand side runs over 6 possible pairs. It follows that $9 \leq 2(3 + 6\mu^2)$ where μ is the worst case coherence. We immediately get $\mu \geq \frac{1}{2}$. The frame formed by the vectors $\phi_1 = (0, 1)$, $\phi_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $\phi_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ achieves the lower bound, observe that the angles between the vectors are the same.

Problem 2

For a matrix Φ, prove that $\mathrm{spark}(\Phi) \leq \mathrm{rank}(\Phi) + 1$

Solution 2

By definition of rank, every collection of vectors of size strictly larger than rank(Φ) are linearly dependent, so for every matrix Φ, any collection of rank(Φ) + 1 vectors is linearly dependent. By the minimality in the definition of spark, spark(Φ) $\leq$ rank(Φ) + 1.

Problem 3

Consider a tight frame $\{\phi_1, \dots, \phi_m\} \in \mathbb{C}^d$ with frame bound A.

(a) Prove that the associated frame operator S is equal to $c(A)I_d$, where $c(A)$ is a constant that depends only on A and I_d is the $d \times d$ identity matrix.

(b) Prove that if $A = 1$ and all vectors have unit norm, then the frame must be an orthonormal basis.

(c) Can you give an example of a tight frame that is not an orthogonal basis?

Solution 3

(a) Let Φ be a matrix whose columns are $\phi_1, \dots, \phi_m$, observe that $\sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2 = \|\Phi^* x\|_2^2$ and since the frame is tight with frame bound A, $\|\Phi^* x\|_2^2 = \sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2 = A\|x\|_2^2$. This implies that Φ^* is a scaled isometry because $\frac{1}{\sqrt{A}}\Phi^*$ is an isometry. It follows that $\langle x, x \rangle = \frac{1}{A}\langle \Phi^* x, \Phi^* x \rangle = \frac{1}{A}\langle \Phi\Phi^* x, x \rangle$. The equality holds for all vectors $x \in \mathbb{C}^d$, it follows that $\frac{1}{A}\Phi\Phi^*$ is the identity matrix. By definition of the frame operator, $\mathrm{S} = AI_d$. The proof reveals more than it, we can use the same argument to show that the converse is also true.

(b) By the previous item we know that $\langle \mathrm{S}\phi_i, \phi_i \rangle = 1$ because $A = 1$ and all ϕ_i's are unitary. We expand the first term of the equality and get

$$1 = \langle \mathrm{S}\phi_i, \phi_i \rangle = \sum_{k=1}^{m} |\langle \phi_k, \phi_i \rangle|^2 = \|\phi_i\|_2^2 + \sum_{k \neq i} |\langle \phi_k, \phi_i \rangle|^2$$
$$= 1 + \sum_{k \neq i} |\langle \phi_k, \phi_i \rangle|^2$$

It follows that $\sum_{k \neq i} |\langle \phi_k, \phi_i \rangle|^2 = 0$, so the vectors in frame are unitary and pairwise orthogonal. Every frame is a spanning set (Homework 4), so the frame must be an orthonormal basis.

(c) Consider the frame $\{(\sqrt{3}, 0); (-1, \sqrt{2}); (-1, \xi\sqrt{2}); (-1, \xi^2\sqrt{2})\}$ where $\xi = e^{\frac{2\pi i}{3}}$. It is easy to see that is a tight frame and it is not an orthogonal basis because there are four vectors in a two dimensional space.