Ex 2

Claim $\hom_{\mathbb{Z}}(G,\mathbb{Z}) \otimes_{\mathbb{Z}} R \cong \hom_{\mathbb{Z}}(G,R)$ if G is a free group of finite rank.

Proof $G \cong \mathbb{Z}^n$, $n \geq 0$

$$\begin{aligned}
\hom_{\mathbb{Z}}(G,\mathbb{Z}) \otimes_{\mathbb{Z}} R &\cong \hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}) \otimes_{\mathbb{Z}} R \\
&\cong \bigoplus_{i=1}^{n} \hom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} R \\
&\cong \left(\bigoplus_{i=1}^{n} \mathbb{Z}\right) \otimes_{\mathbb{Z}} R \\
&\cong \bigoplus_{i=1}^{n} R
\end{aligned}$$

$$\hom_{\mathbb{Z}}(G,R) \cong \hom_{\mathbb{Z}}(\mathbb{Z}^n,R) \cong \bigoplus_{i=1}^{n} \hom(\mathbb{Z},R) \cong \bigoplus_{i=1}^{n} R.$$

(An explicit isomorphism:

$$\begin{aligned}
\hom_{\mathbb{Z}}(G,\mathbb{Z}) \otimes_{\mathbb{Z}} R &\cong \hom_{\mathbb{Z}}(G,R) \\
\varphi \otimes r &\longmapsto (g \mapsto \varphi(g)\cdot r)
\end{aligned}$$
)

$\square$

⚠ Not true for infinite rank!

Basically because

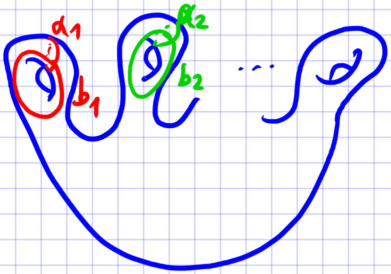
$$\left(\prod_{i\in I} \mathbb{Z}\right) \otimes_{\mathbb{Z}} R \longrightarrow \prod_{i\in I} R$$

$$(n_i)_{i\in I} \otimes r \longmapsto (n_i \cdot r)_{i\in I}$$

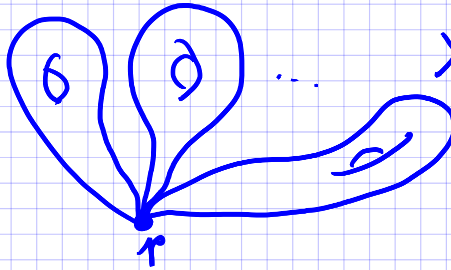
is <u>not</u> surjective in general for infinite I.

Ex 4

Σ_g



$\xrightarrow{\pi}$



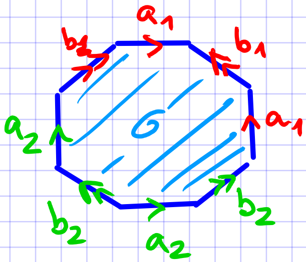
X

Question What is

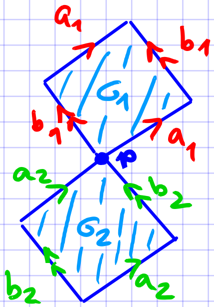
$$\pi_* : H_2(\Sigma_g) \longrightarrow H_2(X) \cong H_2(T^2) \oplus \dots \oplus H_2(T^2) \quad ?$$

Use cellular homology!

Σ_g



$\xrightarrow{\pi}$



X

$$H_2(\Sigma_g) \cong \mathbb{Z}G \qquad H_2(X) \cong \underbrace{\mathbb{Z}G_1}_{\cong H_2(T^2)} \oplus \underbrace{\mathbb{Z}G_2}_{\cong H_2(T^2)}$$

π is cellular (sends n-skeleton to n-skeleton)

$\Rightarrow$ can compute

$$\begin{array}{cccc} \pi_{cw} : & C_2^{cw}(\Sigma_g) & \longrightarrow & C_2^{cw}(X) \\ & \| & & \| \\ & \mathbb{Z}G & & \mathbb{Z}G_1 \oplus \mathbb{Z}G_2 \\ & G & \longmapsto & (G_1, G_2) \end{array}$$

<u>Ex5</u> Hints : • Recall the definition of

$$\delta : H^*(A;R) \longrightarrow H^*(X,A;R) :$$

For $a \in H^p(A;R)$ choose $\varphi \in S^p(A;R)$ with $\delta\varphi = 0$, $[\varphi] = a$.

Choose $\hat{\varphi} \in S^p(X;R)$ with $\hat{\varphi}|_{S_p(A)} = \varphi$.

Then $\delta\hat{\varphi} \in S^{p+1}(X;R)$ satisfies

$$(\delta\hat{\varphi})|_{S_p(A)} = \delta\varphi = 0,$$

So that we can view $\delta\hat{\varphi} \in S^{p+1}(X,A;R)$. It's a cocycle.

Put

$$\delta(a) := [\delta\hat{\varphi}] \in H^{p+1}(X,A;R).$$

- Recall that $a \times b = [pr_1^* \varphi \cup pr_2^* \psi]$

  for $a = [\varphi] \in H^k(X)$, $b = [\psi] \in H^l(Y)$,

  $pr_1 : X \times Y \longrightarrow X$, $pr_2 : X \times Y \longrightarrow Y$.