⚠ "$\mathrm{Ext}(H;k) = 0$ for a field k" is WRONG!!

<u>Example</u> $\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

<u>Wrong "proof"</u> $\dots \to F_1 \xrightarrow{f_2} F_0 \xrightarrow{\varepsilon} H \to 0$ free resolution of abelian group H

e.g. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$

$$\Rightarrow \quad \hom_{\mathbb{Z}}(F_2,k) \leftarrow \hom_{\mathbb{Z}}(F_1,k) \longleftarrow \hom_{\mathbb{Z}}(F_0,k) \longleftarrow \hom_{\mathbb{Z}}(H,k) \leftarrow 0$$

~~exact because k is a field~~ $\Rightarrow$ $\mathrm{Ext}(H;k) = H^1(F_\bullet;k) = 0$.

For above Example:

$$\begin{tikzcd}
0 & \hom(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \arrow[l] \arrow[d, phantom, "\cong"] & \hom(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \arrow[l] \arrow[d, phantom, "="] & \hom(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \arrow[l] & 0 \arrow[l] \\
& \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \arrow[l, "\times 2 = 0"'] & &
\end{tikzcd}$$

not exact!

$$\mathrm{Ext}(H;k) = \ker(\mathbb{Z}/2\mathbb{Z} \to 0) \big/ \mathrm{im}(\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

Rmk There is a generalization for R-modules A, B instead of abelian groups H, G:

$$\mathrm{Ext}^i_R(A,B).$$

It holds that $\mathrm{Ext}^i_k(A,k) = 0$ for $i>0$ for k = field

Computation of $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G)$:

Free resolution for $\mathbb{Z}/n\mathbb{Z}$: $0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$

Dualize:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \hom(\mathbb{Z},G) & \longleftarrow & \hom(\mathbb{Z},G) & \longleftarrow & \hom(\mathbb{Z}/n\mathbb{Z},G) & \longleftarrow & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
 & & G & \xleftarrow{\times n} & G & \longleftarrow & \{g\in G \mid ng=0\} & \longleftarrow & 0
\end{array}$$

$\varphi \mapsto \varphi(1)$

Compute cohomology here!

$$\Longrightarrow \mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}; G) \cong \ker(G \longrightarrow 0) \big/ \mathrm{im}(G \xrightarrow{\times n} G) \cong G/nG.$$