Ex 1 Solutions added.

Ex 5 Bilinear maps

$$A \times B \longrightarrow C$$

of R-modules are equivalent to linear maps

$$A \otimes_R B \longrightarrow C.$$

For example, $H^*(X;R)$, $H^*(Y;R)$ are R-modules (and also $\mathbb{Z}$-modules) and so we can view the cross product as

$$\times : H^*(X;R) \times H^*(Y;R) \longrightarrow H^*(X\times Y;R)$$

or

$$\times : H^*(X;R) \otimes_{\mathbb{Z}} H^*(Y;R) \longrightarrow H^*(X\times Y;R)$$

or

$$\times : H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X\times Y;R)$$

Ex 8 * $H^{2k}(\mathbb{C}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0,\dots,n \\ 0 & \text{else} \end{cases}$

If $\alpha \in H^2(\mathbb{C}P^n;\mathbb{Z})$ denotes a generator, then

$$\underbrace{\alpha \cup \dots \cup \alpha}_{k\text{-times}} \in H^{2k}(\mathbb{C}P^n;\mathbb{Z}), \quad 1 \le k \le n$$

is a generator of $H^{2k}(\mathbb{C}P^n;\mathbb{Z})$.

In particular, $\alpha \cup \alpha \neq 0$ in $H^*(\mathbb{C}P^2;\mathbb{Z})$ for a generator of $H^2(\mathbb{C}P^2;\mathbb{Z})$.

Iso of rings: $$\begin{array}{ccc} H^*(\mathbb{C}P^2;\mathbb{Z}_p) & \cong & H^*(\mathbb{C}P^2;\mathbb{Z}) \otimes \mathbb{Z}_p \\ \cup & & \cup \\ i^*(\alpha) & \hat{=} & \alpha \otimes 1 \end{array}$$, α as above.

$\Rightarrow$ $$i^*(\alpha \cup \alpha) = i^*(\alpha) \cup i^*(\alpha) = (\alpha \otimes 1) \cup (\alpha \otimes 1) = (\alpha \cup \alpha) \otimes 1 \neq 0$$

$\Rightarrow$ $\alpha \cup \alpha \neq 0$.

* $$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1,2 \\ \mathbb{Z}_p & i=3 \\ \mathbb{Z} & i=4 \end{cases}$$

Cup product of elements in $H^3(X;\mathbb{Z})$, $H^4(X;\mathbb{Z})$ are zero for degree reasons.

* $$\tilde{H}^*(Y;\mathbb{Z}_p) = \mathbb{Z}_p\beta \oplus \mathbb{Z}_p\beta' \oplus \mathbb{Z}_p\gamma$$

(β generator in degree 2,
β' generator in degree 3,
γ generator in degree 4)

ring iso

$$\cong \underbrace{\tilde{H}^*(M(\mathbb{Z}_p;2);\mathbb{Z}_p)}_{=\mathbb{Z}_p\beta \oplus \mathbb{Z}_p\beta'} \oplus \underbrace{\tilde{H}^*(S^4;\mathbb{Z}_p)}_{=\mathbb{Z}_p\gamma}$$

componentwise product

$$\Rightarrow \quad \beta \cup \beta \in H^4(M(\mathbb{Z}_p;2);\mathbb{Z}_p) = 0$$

$$\Rightarrow \quad \beta \cup \beta = 0$$

Ex 11 $\alpha \in H^*(X)$, $\alpha' \in H^*(X)$, $\beta \in H^*(X,A)$, $\gamma \in H^*(A)$

$H(X)$-module via $H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)$

$H^*(X)$-module via $H^*(X) \otimes H^*(A) \longrightarrow H^*(A)$, $\alpha \otimes \beta \longmapsto i^*(\alpha) \cup \beta$

$H^*(X)$-module via $H^*(X) \otimes H^*(X,A) \xrightarrow{\cup} H^*(X,A)$

$$H^*(X,A) \xrightarrow{j^*} H^*(X) \xrightarrow{i^*} H^*(A) \xrightarrow{\delta^*} H^{*+1}(X,A)$$

$$j^*(\alpha \cup \beta) = \alpha \cup j^*(\beta) \qquad \delta^*(i^*(\alpha) \cup \gamma) = \alpha \cup \delta^*(\gamma)$$

$$i^*(\alpha \cup \alpha') = i^*(\alpha) \cup i^*(\alpha')$$

$\Longrightarrow$ j^*, i^*, δ^* are $H^*(X)$-module homomorphisms.

Ex 12 $\quad H^*(T^n;\mathbb{Z}) \cong H^*(S^1;\mathbb{Z}) \otimes \cdots \otimes H^*(S^1;\mathbb{Z})$

follows from proof of Thm 3.2. in Chapter VI.3 in Bredon, because $H^*(S^1;\mathbb{Z})$ is a finitely generated **free** abelian group.

n=2: $\quad H^*(T^n;\mathbb{Z}) \underset{\Phi}{\overset{\cong}{\longleftarrow}} H^*(S^1;\mathbb{Z}) \otimes H^*(S^1;\mathbb{Z})$

is a ring homomorphism, because $\quad (a, a', b, b' \in H^*(S^1;\mathbb{Z}))$

$$\Phi\big((a\otimes b)\cdot(a'\otimes b')\big) = \Phi\big((-1)^{|b||a'|}(a\cup a')\otimes(b\cup b')\big)$$

$$= (-1)^{|b||a'|}(a\cup a')\times(b\cup b')$$

$$= (a\times b)\cup(a'\times b')$$

$$= \Phi(a\otimes b)\cup\Phi(a'\otimes b').$$

product on the right side

Claim

$$\mathbb{Z}[\alpha_1]/\langle \alpha_1^2 = 0\rangle \otimes \mathbb{Z}[\alpha_2]/\langle \alpha_2^2 = 0\rangle \cong \mathbb{Z}[\beta_1,\beta_2]/\langle \beta_1^2 = \beta_2^2 = 0,\ \beta_1\beta_2 = -\beta_2\beta_1\rangle$$

Proof

$$\mathbb{Z}[\alpha_1]/\langle \alpha_1^2 = 0\rangle \otimes \mathbb{Z}[\alpha_2]/\langle \alpha_2^2 = 0\rangle \longleftarrow \mathbb{Z}[\beta_1,\beta_2]/\langle \beta_1^2 = \beta_2^2 = 0\rangle$$

$$\begin{aligned} \alpha_1 \otimes 1 &\xleftarrow{\ \psi\ } \beta_1 \\ 1 \otimes \alpha_2 &\longleftarrow \beta_2 \end{aligned}$$

kernel:
$$\begin{aligned} \psi(\beta_1\beta_2) = \psi(\beta_1)\psi(\beta_2) &= (\alpha_1 \otimes 1)(1 \otimes \alpha_2) \\ &= \alpha_1 \otimes \alpha_2 \\ &= -(1 \otimes \alpha_2)(\alpha_1 \otimes 1) \\ &= -\psi(\beta_2)\psi(\beta_1) \\ &= \psi(-\beta_2\beta_1) \end{aligned}$$

$\Rightarrow \quad \ker \psi = \langle \beta_1\beta_2 + \beta_2\beta_1 \rangle.$ $\square$