Question: Why do we use the Koszul sign convention?

E.g.

- $$f \otimes g\,(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$$

  ($f: A \longrightarrow A'$, $g: B \longrightarrow B'$ graded homomorphisms of graded rings)

- $$\delta\varphi\,(a) = (-1)^{|a|} \varphi(\partial a)$$

  ($a \in S_{p+1}(X)$, $\varphi \in S^p(X;R)$)

Answer: 1) We can't do without signs! See next pages.

2) Koszul sign convention yields formulas one can guess by applying the rule

"Whenever two elements of degrees m and n are interchanged, insert the sign $(-1)^{mn}$."

Two examples where signs can't be avoided:

A) Tensor product of chain complexes:

Let $(A_\cdot, \partial_A)$, $(B_\cdot, \partial_B)$ be chain complexes. The formula

$$\delta(a\otimes b) := \partial_A a \otimes b + a \otimes \partial_B b$$

does <u>not</u> define a differential on $A\otimes B$!

Indeed,

$$\delta^2(a\otimes b) = \underbrace{\partial_A^2 a}_{=0} \otimes b + \underset{\substack{\| \\ (\partial_A\otimes id)(a\otimes\partial_B b)}}{\partial_A a \otimes \partial_B b} + \underset{\substack{\| \\ (id\otimes\partial_B)(\partial_A a\otimes b)}}{\partial_A a \otimes \partial_B b} + a\otimes \underbrace{\partial_B^2 b}_{=0}$$

$$\neq 0.$$

these terms need to have different signs!

Try to insert signs in the definition of δ, to achieve $\delta^2=0$! You will see, that you need a sign that depends on the degree of a or b.

<u>One</u> possible way is

$$\tilde{\delta}(a\otimes b) := \partial_A a \otimes b + (-1)^{|a|}\, a\otimes \partial_B b$$

So with the Koszul convention one gets

$$\tilde{\delta} = \partial_A \otimes id + id \otimes \partial_B .$$

B) Hom of chain complexes

Let $(A_\cdot, \partial_A)$, $(B_\cdot, \partial_B)$ be chain complexes. Define

$$\hom^p(A_\cdot, B_\cdot) := \prod_{i \in \mathbb{Z}} \hom(A_i, B_{i-p}).$$

We would like to define a differential

$$\tilde{\delta} : \hom^p(A_\cdot, B_\cdot) \longrightarrow \hom^{p+1}(A_\cdot, B_\cdot)$$

"compatible" with ∂_A and ∂_B.

Compatibility should look like a Leibniz formula:

$$\partial_B \langle f, a \rangle = \langle \tilde{\delta} f, a \rangle + \langle f, \partial_A a \rangle.$$

But then:

$$0 = \partial_B^2 \langle f, a \rangle = \langle \tilde{\delta}^2 f, a \rangle + \underbrace{\langle \tilde{\delta} f, \partial a \rangle + \langle \tilde{\delta} f, \partial a \rangle}_{\neq 0} + \langle f, \underbrace{\partial_A^2 a}_{=0} \rangle$$

and hence $\tilde{\delta}^2 \neq 0$!

Again we need to introduce signs. One way:

$$\partial_B \langle f, a\rangle = \langle \tilde{\delta} f, a\rangle + (-1)^{|f|}\langle f, \partial_A a\rangle .$$

<u>Special case</u>: $A_\bullet = S_\bullet(X)$, $B_\bullet = (\dots \to 0 \to \underset{\substack{\uparrow \\ \text{degree } 0}}{R} \to 0 \dots)$

Then

$$S^p(X;R) = \mathrm{Hom}^p(A_\bullet, B_\bullet)$$

and for $a \in S_{p+1}(X)$, $\varphi \in S^p(X;R)$:

$$0 = \partial_B \underbrace{\langle \varphi, a\rangle}_{=0} = \langle \delta\varphi, a\rangle + (-1)^{|\varphi|}\langle \varphi, \partial a\rangle$$

$$\Longrightarrow \quad \delta\varphi(a) = -(-1)^p \, \varphi(\partial a)$$

$$= (-1)^{p+1}\, \varphi(\partial a)$$

$$= (-1)^{|a|}\, \varphi(\partial a).$$