## Problem set 2

- 1. Let H, H', H'' and G be Abelian groups and  $f: H \to H', g: H' \to H''$  group homomorphisms. Show that f induces a well defined homomorphism  $f_{\text{Tor}}: \text{Tor}(H, G) \to \text{Tor}(H', G)$ . Moreover show that  $\text{id}_{\text{Tor}} = \text{id}, (g \circ f)_{\text{Tor}} = g_{\text{Tor}} \circ f_{\text{Tor}}$  and if f is an isomorphism then  $(f^{-1})_{\text{Tor}} = (f_{\text{Tor}})^{-1}$ .
- 2. Prove that the sequence in the universal coefficient theorem for homology is natural with respect to chain maps. That is, given a chain map  $f: C_* \to D_*$  show that the diagram

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \operatorname{Tor}(H_{n-1}(C),G) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow f_* \qquad \qquad \downarrow$$

$$0 \longrightarrow H_n(D) \otimes G \longrightarrow H_n(D;G) \longrightarrow \operatorname{Tor}(H_{n-1}(D),G) \longrightarrow 0$$

commutes.

Remark: The statement also holds for the universal coefficient theorem for cohomology.

- 3. Let  $C_*, D_*$  be chain complexes of free Abelian groups and assume that  $f: C_* \to D_*$  is a quasi-isomorphism, i.e. a chain map such that  $f_*: H_*(C) \to H_*(D)$  is an isomorphism. Let G be an Abelian group. Prove the following statements using naturality of the sequences in the universal coefficient theorems.
  - (a)  $f \otimes id : C_* \otimes G \to D_* \otimes G$  is a quasi-isomorphism.
  - (b)  $f^* : \text{Hom}(D_*, G) \to \text{Hom}(C_*, G)$  is a quasi-isomorphism.
- 4. Show that the splitting  $H^n(X;G) \cong \operatorname{Ext}(H_{n-1}(X);G) \oplus \operatorname{Hom}(H_n(X);G)$  whose existence is asserted by the universal coefficient theorem for cohomology *cannot* be natural in X.

  Hint: Consider the map  $\phi: \mathbb{R}P^2 \to S^2$  given by collapsing  $\mathbb{R}P^1 \subset \mathbb{R}P^2$  to a point.
- 5. The Klein bottle K has  $H_0(K; \mathbb{Z}) \cong \mathbb{Z}$ ,  $H_1(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$  and all other homology groups vanish. Use this to compute the cohomology of K with coefficients in  $\mathbb{Z}$  and the cohomology and homology with coefficients in  $\mathbb{Z}_p$  for p prime.
- 6. Let X be a topological space and let  $A, B \subset X$  be subsets. Denote by  $S_k(A+B) \subset S_k(X)$  the subspace of chains which are sums of simplices entirely contained in A or B. Show that the quotient  $S_k(X)/S_k(A+B)$  is free.
- 7. Let X be a topological space. Show that

$$H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z}) \otimes \mathbb{Q}.$$

and

$$H^n(X; \mathbb{Z}) \cong \operatorname{Hom}(H_n(X; \mathbb{Z}), \mathbb{Q}).$$

*Hint:* Show that  $Tor(A, \mathbb{Q}) = 0$  and  $Ext(A, \mathbb{Q}) = 0$  for any abelian group A and use the universal coefficients theorems.

8. Let A be an abelian group and R a commutative ring. View  $A \otimes_{\mathbb{Z}} R$  as an R-module in the obvious way. Consider also  $\hom_{\mathbb{Z}}(A,R)$  and  $\hom_{R}(A \otimes_{\mathbb{Z}} R,R)$  and view them as R-modules in the obvious way.

(a) Show that there exists an isomorphism

$$\varphi \colon \hom_{\mathbb{Z}}(A, R) \xrightarrow{\cong} \hom_{R}(A \otimes_{\mathbb{Z}} R, R)$$

- of R-modules, which is natural wrt homomorphisms of abelian groups  $A \to A'$ .
- (b) Let  $C_{\bullet}$  be a chain complex of abelian groups and consider the cochain complexes  $\hom_{\mathbb{Z}}(C_{\bullet}, R)$  and  $\hom_{R}(C_{\bullet} \otimes_{\mathbb{Z}} R, R)$ . Show that the (co-)boundary operators of these cochain complexes are R-linear (so these are cochain complexes of R-modules). Show that the isomorphism  $\varphi$  from (a) can be chosen in this case to be a cochain isomorphism

$$\varphi \colon \hom_{\mathbb{Z}}(C_{\bullet}, R) \xrightarrow{\cong} \hom_{R}(C_{\bullet} \otimes_{\mathbb{Z}} R, R)$$