Solutions to problem set 2

1. Let H, H' be Abelian groups with free resolutions $F \to H$, $F' \to H'$. By the free resolution lemma, we can extend any given group homomorphism $f : H \to H'$ to a chain map $\widetilde{f} : F \to F'$. Recall that by definition we have $\mathrm{Tor}(H, G) = H_1(F \otimes G)$ and $\mathrm{Tor}(H', G) = H_1(F' \otimes G)$, and so we define the action of $\mathrm{Tor}(-, G)$ on f by

$$f_{\mathrm{Tor}} := (\widetilde{f} \otimes \mathrm{id})_* : H_1(F' \otimes G) \to H_1(F \otimes G).$$

This is independent of the choice of lift $\widetilde{f}$ as that is unique up to chain homotopy. To see that this makes $\mathrm{Tor}(-, G)$ a functor, note that $\mathrm{id}_{\mathrm{Tor}} = \mathrm{id}$ because we can take as a lift of $\mathrm{id} : H \to H$ simply id of any free resolution of H. Moreover, $(fg)_{\mathrm{Tor}} = g_{\mathrm{Tor}} f_{\mathrm{Tor}}$, because if $\widetilde{f}$ lifts f and $\widetilde{g}$ lifts g, then $\widetilde{g}\widetilde{f}$ lifts gf.

The case of $\mathrm{Ext}(-, G)$ is analogous. (Of course, these are are just special cases of how in general one constructs the action of derived functors on morphisms.)

2. We discuss the sequence $0 \to H_n(C) \to H_n(C \otimes G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$ appearing in the universal coefficient theorem for homology. Recall that we constructed this as

$$0 \to \mathrm{coker}(i_n \otimes \mathrm{id}) \to H_n(C; G) \to \ker(i_{n-1} \otimes \mathrm{id}) \to 0 \tag{1}$$

with $i_n : B_n \to Z_n$ the inclusion map, and then noted that

$$\mathrm{coker}(i_n \otimes \mathrm{id}) \cong H_n(C) \otimes G \quad \text{and} \quad \ker(i_{n-1} \otimes \mathrm{id}) \cong \mathrm{Tor}(H_{n-1}(C), G). \tag{2}$$

It is clear that a chain map $\phi : C \to C'$ induces a morphism of short exact sequences between (1) and its counterpart for C' (just think about how we arrived at (1)). Moreover, one checks easily that under the identifications (2) and the corresponding ones for C', the outer maps in this morphism of SES are $\phi_* : H_n(C) \to H_n(C')$ and $(\phi_*)_{\mathrm{Tor}}$.

3. (a) Naturality of the short exact sequence in the universal coefficient theorem for homology says that the diagram

$$\begin{tikzcd} 0 \arrow[r] & H_n(C) \otimes G \arrow[r] \arrow[d, "f_* \otimes \mathrm{id}"] & H_n(C; G) \arrow[r] \arrow[d, "f_*"] & \mathrm{Tor}(H_{n-1}(C), G) \arrow[r] \arrow[d, "(f_*)_{\mathrm{Tor}}"] & 0 \\ 0 \arrow[r] & H_n(D) \otimes G \arrow[r] & H_n(D; G) \arrow[r] & \mathrm{Tor}(H_{n-1}(D), G) \arrow[r] & 0 \end{tikzcd}$$

commutes. The outer two maps are isomorphisms because $f_* : H_*(C) \to H_*(D)$ is an isomorphism by assumption and by functoriality of $\mathrm{Tor}(-, G)$. Hence $f_* : H_*(C; G) \to H_*(D; G)$ is an isomorphism by the 5-lemma.

(b) Same argument as in (a) using the universal coefficient theorem for cohomology.

4. Consider the diagram

$$\begin{tikzcd} H^2(S^2; G) \arrow[r] \arrow[d, "\phi^*"'] & \mathrm{Ext}(H_1(S^2), G) \oplus \mathrm{Hom}(H_2(S^2), G) \arrow[d, "(\phi_*)^{\mathrm{Ext}} \oplus (\phi_*)^*"] \\ H^2(\mathbb{R}P^2; G) \arrow[r] & \mathrm{Ext}(H_1(\mathbb{R}P^2), G) \oplus \mathrm{Hom}(H_2(\mathbb{R}P^2), G) \end{tikzcd}$$

Note that we have $\operatorname{Ext}(H_1(S^2), G) = 0$ and $\operatorname{Hom}(H_2(\mathbb{R}P^2), G) = 0$ because $H_1(S^2) = 0$, $H_2(\mathbb{R}P^2) = 0$, and hence the map on the right vanishes for every Abelian group G. If the splitting were natural, the map $\phi^* : H^2(S^2; G) \to H^2(\mathbb{R}P^2; G)$ would consequently also have to vanish for every G.

We will show, in contrast, that $\phi^* : H^2(S^2; \mathbb{Z}_2) \to H^2(\mathbb{R}P^2; \mathbb{Z}_2)$ is an isomorphism. To see this, note that $\phi : \mathbb{R}P^2 \to S^2$ is a cellular map with respect to the usual CW complex structures of $\mathbb{R}P^2$ (with one cell in each degree $0, 1, 2$) and S^2 (with one cell in degree 0 and one in degree 2). The map induced by ϕ on cellular chains takes the generator corresponding to the unique 2-cell of $\mathbb{R}P^2$ to the generator corresponding to the unique 2-cell of S^2 (recall the description of this map!). Dualizing, this implies that the map induced by ϕ on the cellular cochain complexes with coefficients in $\mathbb{Z}_2$ looks as follows:

$$\begin{tikzcd}
0 & \mathbb{Z}_2 \arrow[l] & \mathbb{Z}_2 \arrow[l, "0"'] & \mathbb{Z}_2 \arrow[l, "0"'] & 0 \arrow[l] \\
0 & \mathbb{Z}_2 \arrow[l] \arrow[u, "\cong"] & 0 \arrow[l] \arrow[u] & \mathbb{Z}_2 \arrow[l] \arrow[u, "\cong"] & 0 \arrow[l]
\end{tikzcd}$$

In particular, the induced map $H^2(S^2; \mathbb{Z}_2) \to H^2(\mathbb{R}P^2; \mathbb{Z}_2)$ is an isomorphism.

5. The universal coefficient theorem for homology tells us that there is a splitting

$$H_n(K; G) \cong (H_n(K) \otimes G) \oplus \operatorname{Tor}(H_{n-1}(K), G)$$

for every Abelian group G. We have $H_0(K) \otimes \mathbb{Z}_p = \mathbb{Z}_p$ and $H_1(K) \otimes \mathbb{Z}_p = \mathbb{Z}_p \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_p)$; note that $\mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_p = 0$ for odd p (which doesn't have to be prime for that; in general, $\mathbb{Z}_q \otimes \mathbb{Z}_{q'} = 0$ if q, q' are coprime, as $1 = qm + q'm'$ for certain $m, m' \in \mathbb{Z}$, from which it follows that $1 \otimes 1 = 0$ in $\mathbb{Z}_q \otimes \mathbb{Z}_{q'}$). Moreover, $\operatorname{Tor}(H_0(K), \mathbb{Z}_p) = 0$ as $H_0(K)$ is free and $\operatorname{Tor}(H_1(K), \mathbb{Z}_p) = \operatorname{Tor}(\mathbb{Z}_2, \mathbb{Z}_p) = \ker(\mathbb{Z}_p \xrightarrow{2} \mathbb{Z}_p)$, which is $\mathbb{Z}_2$ for $p = 2$ and 0 if p is odd. Combining all that, we obtain

$$H_0(K; \mathbb{Z}_2) = \mathbb{Z}_2, \quad H_1(K; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_2(K; \mathbb{Z}_2) = \mathbb{Z}_2$$

and

$$H_0(K; \mathbb{Z}_p) = \mathbb{Z}_p, \quad H_1(K; \mathbb{Z}_p) = \mathbb{Z}_p, \quad H_2(K; \mathbb{Z}_p) = 0$$

for p odd. All other groups vanish.

From the universal coefficients theorem for cohomology, we obtain a splitting

$$H^n(K; G) \cong \operatorname{Ext}(H_{n-1}(K), G) \oplus \operatorname{Hom}(H_n(K); G)$$

for every Abelian group G. We have $\operatorname{Ext}(H_0(K), G) = 0$ as $H_0(K)$ is free and $\operatorname{Ext}(H_1(K); G) = \operatorname{Ext}(\mathbb{Z}_2, G) \cong G/2G$, which is $\mathbb{Z}_2$ for $G = \mathbb{Z}$ or $G = \mathbb{Z}_2$ and 0 for $G = \mathbb{Z}_p$ with p odd. Moreover, $\operatorname{Hom}(H_0(K); G) = G$, and $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$ implies that

$$\operatorname{Hom}(H_1(K); G) = \begin{cases} \mathbb{Z}, & G = \mathbb{Z} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & G = \mathbb{Z}_2 \\ \mathbb{Z}_p, & G = \mathbb{Z}_p \text{ with } p \text{ odd} \end{cases}$$

It follows that

$$H^0(K; \mathbb{Z}) = \mathbb{Z}, \quad H^1(K; \mathbb{Z}) = \mathbb{Z}, \quad H^2(K; \mathbb{Z}) = \mathbb{Z}_2,$$
$$H^0(K; \mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(K; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H^2(K; \mathbb{Z}_2) = \mathbb{Z}_2$$

and

$$H^0(K; \mathbb{Z}_p) = \mathbb{Z}_p, \quad H^1(K; \mathbb{Z}_p) = \mathbb{Z}_p, \quad H^2(K; \mathbb{Z}_p) = 0$$

for p odd. Again all other groups vanish.

6. $S_k(X)$ splits as $S_k(X) = S_k(A+B) \oplus S_k^{\perp}(A+B)$, where the second summand is generated by all simplices neither contained in A nor in B. Hence the quotient $S_k(X)/S_k(A+B)$ is isomorphic to $S_k^{\perp}(A+B)$, which is free.

7. Let A be an abelian group. We first show that $Tor(A, \mathbb{Q}) = 0$. Choose a free resolution $0 \to F_1 \xrightarrow{i} F_0 \to A \to 0$ and consider the sequence

$$0 \to F_1 \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{i \otimes \mathrm{id}} F_0 \otimes_{\mathbb{Z}} \mathbb{Q} \to A \otimes_{\mathbb{Z}} \mathbb{Q} \to 0.$$

If $i \otimes \mathrm{id}$ is injective, we can deduce that $Tor(A, \mathbb{Q}) = 0$.

In fact, for any injective map $i\colon B \to C$ between abelian groups B and C, the map

$$B \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{i \otimes \mathrm{id}} C \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. Indeed, an element x of $B \otimes \mathbb{Q}$ is of the form $x = \sum b_j \otimes q_j$ with $b_j \in B$, $q_j = \frac{m_j}{n_j}$, $n_j \neq 0$, and the sum is finite. So we can assume that $n_j = n$ for all j and we can write $x = (\sum m_j b_j) \otimes \frac{1}{n}$. If we now assume $i \otimes \mathrm{id}(x) = 0$, we get

$$i\left(\sum m_j b_j\right) \otimes \frac{1}{n} = 0$$

and hence $i(\sum m_j b_j) = 0$. Injectivity of i now yields $\sum m_j b_j = 0$, and so $x = \sum m_j b_j \otimes \frac{1}{n} = 0$. This shows injectivity of $i \otimes \mathrm{id}$.

Remark: Together with Problem 1 from the sheet on tensor products, this shows that $- \otimes_{\mathbb{Z}} \mathbb{Q}$ preserves short exact sequences!

In particular, $\mathrm{Tor}(H_{n-1}(X; \mathbb{Z}), \mathbb{Q}) = 0$ and so the homological universal coefficients theorem implies

$$H_n(X; \mathbb{Q}) \cong H_n(X; \mathbb{Z}) \otimes \mathbb{Q}.$$

For the cohomology the proof is similar. This time, one has to investigate exactness properties of $\hom(-, \mathbb{Q})$. Namely, the following statement will imply $\mathrm{Ext}(A, \mathbb{Q}) = 0$: Let $i\colon B \to C$ be an injective map of abelian groups. Then

$$i^*\colon \hom(C, \mathbb{Q}) \to \hom(B, \mathbb{Q})$$

is surjective.

To prove this, let us view B as a subset of C via i. Let $\varphi \in \hom(B, \mathbb{Q})$. We need to show that φ extends to $\hat{\varphi}\colon C \to \mathbb{Q}$. Let $B \subset C' \subset C$ be the maximal subgroup such that there exists an extension $\varphi'\colon C' \to \mathbb{Q}$. (Use Zorn's lemma to prove existence.) Suppose by contradiction that $C' \neq C$. Then there exists $x \in C \backslash C'$. Moreover, the subgroup $\langle x \rangle \subset C$ generated by x satisfies $\langle x \rangle \cap C' = \{0\}$ because $\mathbb{Q}$ is divisible. Hence we can put $\tilde{\varphi}(x) := q$ for some $q \in \mathbb{Q}$ and extend it linearly to a map $\tilde{\varphi}\colon C' \oplus \langle x \rangle \to \mathbb{Q}$ that extends φ'. This is a contradiction to maximality of C'. We conclude $C' = C$. Surjectivity of i^* now follows.

8. (a) Note that multiplication in R induces a $\mathbb{Z}$-linear map $m\colon R \otimes_{\mathbb{Z}} R \to R$. For $\alpha \in \hom(A, R)$ put $\varphi(\alpha) = m \circ (\alpha \otimes \mathrm{id}) \in \hom_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} R, R)$. Concretely, it is given by

$$\varphi(\alpha)\left(\sum a_j \otimes r_j\right) = \sum \alpha(a_j) r_j$$

for finitely many $a_j \in A$ and $r_j \in R$. In fact, $\varphi(\alpha)$ is R-linear: for $r \in R$

$$\begin{aligned}\varphi(\alpha)\left(r \sum a_j \otimes r_j\right) &= \varphi(\alpha)\left(\sum a_j \otimes r r_j\right) \\ &= \sum \alpha(a_j) r r_j = r \sum \alpha(a_j) r_j \\ &= r\varphi(\alpha)\left(\sum a_j \otimes r_j\right).\end{aligned}$$

This shows that φ is a well-defined $\mathbb{Z}$-linear map

$$\hom_{\mathbb{Z}}(A, R) \longrightarrow \hom_R(A \otimes_{\mathbb{Z}} R, R).$$

It is straightforward to check that it is R-linear and inverse to

$$\psi\colon \hom_R(A \otimes_{\mathbb{Z}} R, R) \to \hom_{\mathbb{Z}}(A, R),\ \psi(\beta)(a) = \beta(a \otimes 1_R)$$

for $\beta \in \hom_R(A \otimes_{\mathbb{Z}} R, R)$ and $a \in A$.

(b) Consider the coboundary operator δ on $\hom_{\mathbb{Z}}(C_\bullet, R)$

$$\begin{aligned} \delta\colon \hom_{\mathbb{Z}}(C_j, R) &\to \hom_{\mathbb{Z}}(C_{j+1}, R) \\ \alpha &\mapsto \alpha \circ \partial, \end{aligned}$$

where ∂ denotes the boundary operator of $C_\bullet$. This is R-linear:

$$\delta(r\alpha) = (r\alpha) \circ \partial = r(\alpha \circ \partial) = r\delta(\alpha).$$

Similarly, the coboundary operator δ_R on $\hom_R(C_\bullet \otimes_{\mathbb{Z}} R, R)$,

$$\begin{aligned} \delta_R\colon \hom_R(C_j \otimes_{\mathbb{Z}} R, R) &\to \hom_R(C_{j+1} \otimes_{\mathbb{Z}} R, R) \\ \beta &\mapsto \beta \circ (\partial \otimes \mathrm{id}), \end{aligned}$$

is R-linear. φ is a cochain isomorphism because the following diagram commutes:

$$\begin{tikzcd} \hom_{\mathbb{Z}}(C_j, R) \arrow[r, "\delta"] \arrow[d, "\varphi"', "\cong"] & \hom_{\mathbb{Z}}(C_{j+1}, R) \arrow[d, "\varphi"', "\cong"] \\ \hom_R(C_j \otimes_{\mathbb{Z}} R, R) \arrow[r, "\delta_R"] & \hom_R(C_{j+1} \otimes_{\mathbb{Z}} R, R). \end{tikzcd}$$

Let's check that it commutes: For $\alpha \in \hom_{\mathbb{Z}}(C_j, R)$ we have

$$\varphi \circ \delta(\alpha) = \varphi(\alpha \circ \delta) = m \circ (\alpha \circ \delta \otimes \mathrm{id})$$

and

$$\begin{aligned} \delta_R \circ \varphi(\alpha) = \delta_R(m \circ (\alpha \otimes \mathrm{id})) &= m \circ (\alpha \otimes \mathrm{id}) \circ (\partial \otimes \mathrm{id}) \\ &= m \circ (\alpha \circ \partial \otimes \mathrm{id}). \end{aligned}$$