D-MATH
FS 2021
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Exercise sheet 1

Probabilistic Number Theory

1 a. The Möbius function $\mu : \mathbb{N} \to \{\pm 1, 0\}$ is defined as

$$\mu(n) := \begin{cases} 0, & \text{if } n \text{ is not squarefree,} \\ (-1)^l, & \text{if } n = p_1 \cdots p_l \text{ is a product of distinct primes.} \end{cases}$$

Show that

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Use the inclusion-exclusion principle.

b. Recall that the Euler totient function $\phi : \mathbb{N} \to \mathbb{N}$ is defined as

$$\phi(n) := |\{1 \leq k \leq n : (k, n) = 1\}|.$$

Show that for all $n \geq 1$ we have

$$\phi(n) = n \sum_{d \mid n} \frac{\mu(d)}{d}.$$

2 Given two functions $f, g : \mathbb{N} \to \mathbb{C}$ we can define their Dirichlet convolution $f * g : \mathbb{N} \to \mathbb{C}$ via

$$(f * g)(n) := \sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right).$$

a. Show that for all $f, g, h : \mathbb{N} \to \mathbb{C}$ we have

$$f * g = g * f \qquad \text{and} \qquad (f * g) * h = f * (g * h).$$

b. **Möbius inversion formula**
Let $f : \mathbb{N} \to \mathbb{C}$ be any function. Show that

$$g(n) = \sum_{d \mid n} f(d) \text{ for all } n \in \mathbb{N} \quad \Leftrightarrow \quad f(n) = \sum_{d \mid n} g(d)\, \mu\left(\frac{n}{d}\right) \text{ for all } n \in \mathbb{N}.$$

c. Show that for all $n \in \mathbb{N}$ we have

$$n = \sum_{d \mid n} \phi(d).$$

d. Let $f, g : \mathbb{N} \to \mathbb{C}$ be functions, and let $s \in \mathbb{C}$ be such that the associated Dirichlet series

$$F(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}, \qquad G(s) := \sum_{n \geq 1} \frac{g(n)}{n^s}$$

converge absolutely. Show that the Dirichlet series associated to $f * g$ satisfies

$$\sum_{n \geq 1} \frac{(f * g)(n)}{n^s} = F(s)G(s).$$

3 For a given integer $N \geq 1$, consider the probability space

$$\Omega_N := \{1, \ldots, N\}$$

and let X_N, Y_N be independent random variables which are uniformly distributed on Ω_N. The goal of this exercise is to prove that

$$\mathbb{P}[(X_N, Y_N) = 1] \to \frac{6}{\pi^2}$$

as $N \to \infty$.

a. Prove that for any $s \in \mathbb{C}$ with $\Re s > 1$ we have

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

b. Prove that

$$\mathbb{P}[(X_N, Y_N) = 1] \to \frac{1}{\zeta(2)}.$$

c. Show that, in order to prove that $\zeta(2) = \frac{\pi^2}{6}$, it suffices to show that

$$\sum_{n \geq 1} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

d. Prove that

$$I := \int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} \, dx dy = \sum_{n \geq 1} \frac{1}{(2n-1)^2}.$$

e. Compute I, e.g. by substituting

$$u := \arccos\left(\sqrt{\frac{1 - x^2}{1 - x^2 y^2}}\right), \qquad v := \arccos\left(\sqrt{\frac{1 - y^2}{1 - x^2 y^2}}\right),$$

so that $x = \frac{\sin u}{\cos v}$ and $y = \frac{\sin v}{\cos u}$.