D-MATH
FS 2021
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Exercise sheet 2

Probabilistic Number Theory

(1) The von Mangoldt function $\Lambda : \mathbb{N} \to \mathbb{R}$ is defined by

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^\nu \text{ is a prime power } (\nu \geq 1), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover we define, for $x \geq 1$,

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

In this exercise, we will assume the following strong form of the Prime Number Theorem: [1]

$$\psi(x) = x + O\left(\frac{x}{(\log x)^2}\right) \qquad (x \geq 2).$$

a. Show that for any $n \in \mathbb{N}$ we have

$$\sum_{d \mid n} \Lambda(d) = \log n.$$

b. Using the summation by parts formula (Lemma A.1.1), prove that for $x \geq 1$ we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \int_1^x \frac{\psi(t)}{t^2}\, dt + \frac{\psi(x)}{x}.$$

c. Using the above version of the Prime Number Theorem, show that uniformly over $x \geq 2$, we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + A + O\left(\frac{1}{\log x}\right),$$

where

$$A = \int_1^\infty \frac{\psi(t) - t}{t^2}\, dt + 1$$

and where the integral converges absolutely.

[1] One can show that this follows e.g. from

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).$$

d. Prove that, uniformly over $x \geq 3$, we have

$$\sum_{n \leq x} \frac{\Lambda(n) \log n}{n} = \frac{1}{2}(\log x)^2 + O(\log \log x).$$

e. You might have seen before that, uniformly over $x \geq 1$, we have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Use this in combination with a.), c.) and d.) to show that

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 + (A + \gamma) \log x + O(\log \log x).$$

f. Show that, on the other hand, we have

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 + O(1)$$

and conclude that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x - \gamma + O\left(\frac{1}{\log x}\right)$$

as well as

$$\int_1^\infty \frac{t - \psi(t)}{t^2}\, dt = 1 + \gamma.$$

2 For $n \geq 1$ let

$$r(n) = |\{(a, b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}|$$

be the number of representation of n as sum of two squares.

a. Show that $r(n) > 0$ if and only if the following holds:
if $p | n$ with $p \equiv 3 \bmod 4$ then p has even exponent in the factorization of n.
Hint: Use the fact that the ring $\mathbb{Z}[i]$ is a unique factorization domain with norm given by $N(a + ib) = a^2 + b^2$ and consider a generator g of $(\mathbb{Z}/p\mathbb{Z})^\times$, i.e. so that the minumum exponent such that $g^m \equiv 1 \bmod p$ is $m = \phi(p) = p - 1$. Note also the identity $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ac - bd)^2$.

b. Show that $r(n)$ is the convolution

$$\frac{r(n)}{4} = (\chi_4 * \mathbb{1})(n),$$

where $\mathbb{1}(n) = 1$ for all n and

$$\chi_4(n) \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \equiv 1 \mod 4 \\ -1 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Observe that since χ_4 is completely multiplicative, $r(n)/4$ is multiplicative.

c. Prove that

$$r(n) \leq 4\tau(n)$$

for all n, where $\tau(n) = \sum_{d|n} 1$ is the number of divisors of n.

d. As x goes to infinity, one has

$$\sum_{n \leq x} r(n) = \pi x + O(\sqrt{x}).$$

Hint: We are counting the integral points inside the circle of radius 1 centered in 0.

e. One can prove ("Landau's Theorem") that there exists a constant c>0 such that the number $R(x)$ of $n \leq x$ with $r(n) \geq 1$ satisfies

$$R(x) \sim c\frac{x}{\sqrt{\log x}}$$

Using this result, prove that the random variables $X_N : n \mapsto r(n)$ on $\Omega_N = \{1, \ldots, N\}$ converges in law to the zero random variable.