D-MATH
FS 2021
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Exercise sheet 3

Probabilistic Number Theory

1 Let q be a positive integer and let χ be a Dirichlet character mod q. We denote the principal character mod q by

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a,q) = 1 \\ 0 & \text{if } (a,q) > 1. \end{cases}$$

a. Let $Q > q$ be a multiple of q. Define

$$\psi(n) = \begin{cases} \chi(n) & \text{if } (n,Q) = 1 \\ 0 & \text{if } (n,Q) > 1. \end{cases}$$

We say that ψ is a character mod Q *induced* by χ. The character χ is sayed to be *primitive* if it is not induced by any character of modulus $< q$. The smallest modulus f so that χ is induced by a character mod f is called the *conductor* of χ.
In particular, it's easy to see that for every $\chi \neq \chi_0 = \chi_{0,q}$ mod q, there is a unique primitive character χ^* mod f which induces χ and so that

$$\chi(n) = \chi^*(n)\chi_0(n) \quad \forall n.$$

Show that the the following are equivalent:

i. χ is primitive.
ii. If $d|q$ and $d < q$, then there is a c such that $c \equiv 1$ mod d, $(c,d) = 1$, $\chi(c) \neq 1$.
iii. If $d|q$ and $d < q$, then for every integer a,

$$S = \sum_{\substack{n=1 \\ n \equiv a \bmod d}}^{q} \chi(n) = 0.$$

b. Let χ be a character mod q. Define the *Gauss sum*

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(a/q),$$

where $e(a \cdot /q)$ is the *additive character* mod q defined by $n \mapsto e^{\frac{2\pi i a n}{q}}$.
Show that if χ is primitive, then

b_1. $\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{q} \overline{\chi}(a)e(an/q)$.

Hint: Start with the case $(n,q)=1$ and then use iii of part a.

b_2. $|\tau(\chi)| = \sqrt{q}$.

c. Prove the *Pólya-Vinogradov inequality*:
Let χ be a non-principal character mod q. Then

$$\sum_{n\leq x} \chi(n) \ll \sqrt{q}\log q.$$

Hint: One way to prove it is by following the steps:

c_1. Show that it suffices to prove it when $x \leq q$;

c_2. Use the formula b_1;

c_3. Find an upper bound for the function $\sum_{a=1}^{q} \left|\sum_{n\leq x} e(an/q)\right|$.

2 Let $q \geq 1$. Consider the $\phi(q) \times \phi(q)$ matrix A with entries $\chi(a)$, as χ varies over all Dirichlet character mod q and a varies in $(\mathbb{Z}/q\mathbb{Z})^{\times}$. Compute $\det(A)$.
Hint: Consider $A\overline{A}^t$.

3 a. Show that the following is false:

$$\log(\tau(n)) = o\Big(\frac{\log n}{\log\log n}\Big),$$

as $n \to +\infty$, where $\sum_{d|n} 1$.

b. Let $P(x) = \prod_{n\leq x} n^{\log n}$. Show that

$$\sqrt[x]{P(x)} = e^2 x^{\log x - 2}\Big(1 + O\Big(\frac{(\log x)^2}{x}\Big)\Big).$$