D-MATH
FS 2021
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Exercise sheet 5

Probabilistic Number Theory

1 Let $\zeta(s)$ be the Riemann zeta function, defined to be the series

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

for $\Re(s) > 1$. Prove that

a. $\sum_{k\geq 2}(\zeta(k) - 1) = 1$;
b. $\sum_{k\geq 1}(\zeta(2k) - 1)^2 = \frac{3}{4}$.

Hint: Write $\sum_{k\geq 2}(\zeta(k) - 1) = \sum_{k\geq 2}\sum_{j\geq 2}\frac{1}{j^k}$ and then change the order of summing.

2 Let $T > 0$ and $\sigma > 1$. Show that

a. $\int_0^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma)T + O(1)$.
b. In general, for a meromorphic function $f(s)$ on $a \leq \sigma \leq b$ with a finitely many poles in this strip, one defines the *Lindelöf function*

$$\begin{aligned} \mu : [a, b] &\longrightarrow \mathbb{R} \cup \{\pm\infty\} \\ \sigma &\longmapsto \inf\{c \in \mathbb{R} \cup \{\pm\infty\} : |f(\sigma + it)| \ll |t|^c \text{ as } |t| \to +\infty\}. \end{aligned}$$

Denote by $\mu(\sigma)$ the Lindelöf function of the Riemann zeta function. Show that it satisfyies

$$\mu(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq 1 \\ 1/2 - \sigma & \text{if } \sigma \leq 0. \end{cases}$$

Moreover, $0 \leq \mu(\sigma) \leq \frac{1}{2} - \frac{\sigma}{2}$ if $0 \leq \sigma \leq 1$.
Hint: Use the following facts:

- $\mu(\sigma)$ is a convex function, in particular for all $a \leq \sigma \leq b$,

$$\mu(\sigma) \leq \frac{b - \sigma}{b - a}\mu(a) + \frac{\sigma - a}{b - a}\mu(b).$$

- One has $\mu(\sigma) = 0$ for $\sigma > 1$.

- Use the functional equation for the Riemann zeta function

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

  and the Stirling's formula.

c. $\int_{-T}^{T}\zeta\big(\frac{1}{2}+it\big)dt = 2T + O(T^{\frac{1}{4}+\varepsilon})$.

Hint: Write $\int_{-T}^{T}\zeta\big(\frac{1}{2}+it\big)dt = \frac{1}{i}\int_{1/2-iT}^{1/2+iT}\zeta(s)ds$ and move the integration line to $\sigma = 2$. then use b.

3 Show that the Riemann hypothesis is equivalent to the following statement:

$$\sum_{n\leq x}\mu(n) = O(x^{\frac{1}{2}+\varepsilon})$$

for every $\varepsilon > 0$.
Hint: Use the Perron's approximation formula that we saw during the exercise class and the fact that the Riemann hypothesis implies that if $\sigma > 1/2$, then

$$\frac{1}{\zeta(\sigma+it)} \ll |t|^{\varepsilon}$$

for every $\varepsilon > 0$, as $|t| \to +\infty$.
For the other implication, use the dexpliocit formula and use summation by parts to represent $1/\zeta(s)$ in terms of an integral with the partial sums.

4 Recall that the *Mills numbers* are real numbers $A > 1$ with the property that $\lfloor A^{3^n}\rfloor$ is a prime number for every positive integer n. In the following assume the RH.

Show that the existence of such a number, more precisely, that given an infinite subset S of integers, and a real number $c > 1$ there exists a constant $B = B(S, c)$ such that

$$\lfloor B^{c^n}\rfloor \in S \text{ for all } n \geq 1,$$

provided the set S has the property that, for some real number $0 < \theta < 1 - \frac{1}{c}$ and all large enough x, the intersection $[x, x + x^{\theta}] \cap S$ is not empty.
Hint: Since $\theta < 1$, there must be some element of the set S in all those "short" intervals. What about primes in short intervals? Use the PNT and the RH.