D-MATH
FS 2021
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Exercise sheet 6

Probabilistic Number Theory

1 Let $(X_p)_p$ be a sequence of independent random variables all uniformly distributed on $\mathbb{S}^1$. For any positive integer n, let

$$X_n = \prod_{p|n} X_p^{v_p(n)},$$

where $v_p(n)$ is the p-adic valuation of n. For any real numbers $q > 0$ and $x > 1$ and any sequence of complex numbers (a_n), prove that the limit

$$\lim_{T\to+\infty} \frac{1}{T} \int_T^{2T} \Big| \sum_{n\leq x} a_n n^{-it} \Big|^q dt$$

exists, and that it is equal to

$$\mathbb{E}\Big(\Big| \sum_{n\leq x} a_n X_n \Big|^q \Big).$$

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a. For $(a_0, \ldots, a_m) \in \mathbb{C}^{m+1}$, $a_0 \neq 0$, prove that there exists $(b_0, \ldots, b_m) \in \mathbb{C}^{m+1}$ such that we have

$$\exp\Big(\sum_{k=0}^{m} b_k s^k \Big) = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + O(s^{m+1})$$

for $s \in \mathbb{C}$.

Now fix a real number σ with $\frac{1}{2} < \sigma < 1$ and let g be a holomorphic function on $\mathbb{C}$ which does not vanish.

b. For any $\varepsilon > 0$, prove that there exists a real number t and $r > 0$ such that

$$\sup_{|s|\leq r} |\zeta(s + \sigma + it) - g(s)| < \varepsilon \frac{r^k}{k!}.$$

c. Let $n > 1$ be an integer. Prove that there exists $t \in \mathbb{R}$ such that for all integers k with $0 \leq k \leq n - 1$, we have

$$|\zeta^{(k)}(\sigma + it) - g^{(k)}(0)| < \varepsilon.$$

d. Let $n > 1$ be an integer. Prove that the image in $\mathbb{C}^n$ of the map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{C}^n \\ t &\longmapsto (\zeta(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it)) \end{aligned}$$

is dense in $\mathbb{C}$.

e. Using (4), prove that if $n, N \geq 1$ are integers, and $F_0, \ldots, F_N$ are continuous functions $\mathbb{C}^n \to \mathbb{C}$, not identically zero, then the functions

$$\sum_{k=0}^{N} s^k F_k(\zeta(s), \zeta'(s), \ldots, \zeta^{(n-1)}(s))$$

is not identically zero. In particular, the Riemann zeta function satisfies no algebraic differential equation.

3 By using a Perron's formula with error term, one can show that for every $T \geq 2$ and for every fixed $s \in \mathbb{C}$ with $\sigma = \mathrm{Re} s > 0$

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s+w) \frac{x^w}{w} dw + O\Big(x^{-\sigma} + \frac{x^{1-\sigma} \log x}{T}\Big),$$

where $c = 1 - \sigma + \frac{1}{\log x}$.

Show that for every $t \in \mathbb{R} \setminus \{0\}$, t fixed, the sum

$$\sum_{n \leq x} \frac{1}{n^{1+it}}$$

is bounded as $x \to +\infty$, but doesn't converge as $x \to +\infty$.