D-MATH
FS 2021
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Solutions 3

Probabilistic Number Theory

1. a. i⇒ii: since χ is primitive, there exist integers $m \equiv n \bmod d$, $(mn, q) = 1$ so that $\chi(m) \neq \chi(n)$ (if not, χ would be induced by a character mod d). Choose c so that $(c, q) = 1$, $cm \equiv n \bmod q$ and we have ii.

   ii⇒iii: let c be as in ii. As k runs through a complete residue system mod q/d, the numbers $n = ac + kcd$ run through all residues q for which $n \equiv a \bmod d$. Thus,

$$S = \sum_{k=1}^{q/d} \chi(ac + kcd) = \chi(c)S.$$

   Since $\chi(c) \neq 1$, $S = 0$.

   iii⇒i: Take $a = 1$ in iii. Then $\chi(1) = 1$ is one term of S. But $S = 0$, so there must be another term $\chi(n)$ in the sum so that $\chi(n) \neq 0, 1$. But $n \equiv 1 \bmod d$, so χ cannot be induced by any character mod d, for any proper divisor of q.

   b. Let $(n, q) = 1$. Then the map $a \mapsto an$ permutes the residues mod q, hence

$$\begin{aligned}\chi(n)\tau(\overline{\chi}) &= \chi(n) \sum_{a \bmod q} \overline{\chi}(a)e(a/q)\\ &= \chi(n) \sum_{a \bmod q} \overline{\chi}(an)e(an/q)\\ &= \sum_{a \bmod q} \overline{\chi}(a)e(an/q),\end{aligned}$$

   which proves b_1 in the case $(n, q) = 1$.
   If $(n, q) > 1$. Pick m and d coprime, so that $m/d = n/q$. Then

$$\sum_{a \bmod q} \overline{\chi}(a)e(an/q) = \sum_{h=1}^{d} e(hm/d) \sum_{\substack{a=1\\ a\equiv h \bmod d}}^{q} \chi(a).$$

Since d is a proper divisor of q and χ is primitive, by iii the inner sum vanishes, so b_1 holds also in this case.

For b_2, note that by orthogonality,

$$\sum_{n=1}^{q} |\tau(\overline{\chi})|^2 |\chi(n)|^2 = |\tau(\overline{\chi})|^2 \phi(q).$$

On the other hand,

$$\begin{aligned}
\sum_{n=1}^{q} |\tau(\overline{\chi})|^2 |\chi(n)|^2 &= \sum_{n=1}^{q} \Big| \sum_{a \bmod q} \overline{\chi}(a) e(na/q) \Big|^2 \\
&= \sum_{n=1}^{q} \Big(\sum_{a \bmod q} \overline{\chi}(a) e(na/q) \Big) \Big(\sum_{b \bmod q} \chi(b) e(-nb/q) \Big) \\
&= \sum_{a,b \bmod q} \overline{\chi}(a) \chi(b) e(n(a-b)/q) \\
&= \sum_{a \bmod q} |\chi(a)|^2 q = \phi(q) q.
\end{aligned}$$

Therefore $|\tau(\overline{\chi})|^2 = q$ and so $|\tau(\chi)| = |\tau(\overline{\chi})| = \sqrt{q}$.

c. Note that $[x] = Nq + r$ for some $N \geq 0$ and $r < q$. We can then write

$$\sum_{n \leq x} \chi(n) = \sum_{\ell=0}^{N-1} \sum_{n=0}^{q-1} \chi(q\ell + n) + \sum_{m=1}^{r} \chi(Nq + m).$$

By periodicity and orthogonality, we have

$$\sum_{n \leq x} \chi(n) = \sum_{n \leq r} \chi(n).$$

Without loss of generality, we can then assume that $x \leq q$.

If χ is primitive, then $\tau(\overline{\chi}) \neq 0$, so by b_1 we can write

$$\sum_{n \leq x} \chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \sum_{n \leq x} e(an/q).$$

Let f be the q-periodic function

$$f(a) = \sum_{n \leq x} e(an/q).$$

Then

$$\Big|\sum_{n\le x}\chi(n)\Big| \le \frac{1}{|\tau(\overline{\chi})|}\sum_{a=1}^{q}|f(a)| = \frac{1}{\sqrt{q}}\sum_{a=1}^{q}|f(a)|.$$

Note that $\sum_{a=1}^{q}|f(a)| = \sum_{a=1}^{q/2}|f(a)| + \sum_{a=q/2+1}^{q}|f(a)|$ and since $f(q-a) = \overline{f(a)}$, one has

$$\sum_{a=q/2+1}^{q}|f(a)| = \sum_{a=q/2+1}^{q}|f(q-a)| = \sum_{b=0}^{q/2-1}|f(b)| \le \sum_{a=1}^{q/2}|f(a)| + |f(0)|.$$

Then

$$\Big|\sum_{n\le x}\chi(n)\Big| \ll \frac{1}{\sqrt{q}}\Big(\sum_{a=1}^{q/2}|f(a)| + |f(0)|\Big).$$

Since we are assuming $x \le q$, the second summand of the RHS is $\ll q$. For the first one, ler $r = e(a/q)$. The function $\sin\pi\alpha$ is concave for $1 \le \alpha \le 1/2$, so the inequality $\sin\pi\alpha \ge 2\alpha$ holds for $1 \le \alpha \le 1/2$. Thus

$$|f(a)| = \left|\frac{1 - r^{[x]+1}}{1 - r}\right| \le \frac{2}{|1 - r|} = \frac{1}{\sin\frac{\pi a}{q}} \le \frac{q}{2\pi a}.$$

Now use the inequality

$$\sum_{a=1}^{q}\frac{1}{a} \le 1 + \int_1^q \frac{dt}{t} \ll \log q$$

and conclude.

d.

2 One has for all $1 \le i, j \le \phi(q)$,

$$(A\overline{A}^t)_{ij} = \sum_{(a,q)=1}\chi_i(a)\overline{\chi_j}(a) = \begin{cases}\phi(q) & \text{if } i = j\\ 0\text{otherwise.}\end{cases}$$

This means that

$$A\overline{A}^t = \mathbb{I}_{\phi(q)}\cdot\phi(q),$$

where $\mathbb{I}_{\phi(q)}$ is the identity matrix of size $\phi(q)$. It turns out that

$$\det(A\overline{A}^t) = \phi(q)^{\phi(q)}.$$

On the other hand,

$$\det(A\overline{A}^t) = \det(A)\det(\overline{A}^t) = \det(A)\overline{\det(A)} = |\det(A)|^2,$$

and so

$$|\det(A)| = \phi(q)^{\phi(q)/2}.$$

3 a. Consider the sequence of natural numbers of the form

$$n = \prod_{p \leq x} p$$

for $x \geq 1$. Then

$$\tau(n) = \prod_{p \leq x} \tau(p) = 2^{\pi(x)},$$

$$\log \tau(n) = \pi(x) \log 2 \sim \frac{x}{\log x} \log 2$$

as $x \to +\infty$ by the prime number theorem. Also, $\log n = \sum_{p \leq x} \log p = \Theta(x) \sim x$. In particular $\log x \sim \log \log n$. For such n we thus have

$$\log \tau(n) \sim \log 2 \frac{\log n}{\log \log n}$$

as $n \to +\infty$.

b. By taking the logarithm we get

$$\log \sqrt[x]{P(x)} = \frac{1}{x} \sum_{n \leq x} (\log n)^2.$$

By partial summation,

$$\begin{aligned} \sum_{n \leq x} (\log n)^2 &= x(\log x)^2 + O((\log x)^2) - 2 \int_1^x \log y dy + O\Big(\int_1^x \frac{\log y}{y} dy \Big) \\ &= x((\log x)^2 - 2 \log x + 2) + O\Big(\frac{(\log x)^2}{x} \Big). \end{aligned}$$

It follows by apllying $\exp(\cdot)$ on both sides,

$$\begin{aligned} \sqrt[x]{P(x)} &= e^{\log x (\log x - 2)} e^2 \exp \Big(O\Big(\frac{(\log x)^2}{x} \Big) \Big) \\ &= e^2 x^{\log x - 2} \Big(1 + O\Big(\frac{(\log x)^2}{x} \Big) \Big). \end{aligned}$$