D-MATH
FS 2021
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Solutions 4

Probabilistic Number Theory

1. Any character of $\mathbb{R}/\mathbb{Z}$ can be pulled back to a character of $\mathbb{R}$ by composing with the canonical projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Let $\xi : \mathbb{R} \to \mathbb{C}^\times$ be the resulting character of $\mathbb{R}$, i.e. $\xi(t) = \chi(t \mod \mathbb{Z})$. In the above we prove a, given this, we'd have $\xi(t) = e^{2\pi ixt}$ for some $x \in \mathbb{R}$, thus $\chi(t \mod \mathbb{Z}) = e^{2\pi ixt}$. Taking $t = 1$, we get $1 = e^{2\pi ix}$. Therefore $x \in \mathbb{Z}$.

   a. Let $\xi : \mathbb{R} \to \mathbb{C}^\times$ be a continuous homomorphism. Because $\xi(0) = 1$, by continuity $\psi(x_0) \neq 0$ for an appropriate x_0. Fixing such x_0, for all real t we have

$$\begin{aligned}\int_t^{t+x_0} \xi(u)du &= \int_0^{x_0} \xi(u+t)du \\ &= \int_0^{x_0} \xi(u)\xi(t)du \\ &= \xi(t)\int_0^{x_0} \xi(u)du.\end{aligned}$$

   Therefore

$$\xi(t) = \frac{\psi(t+x_0) - \psi(t)}{\psi(x_0)}.$$

   The right side, by the fundamental theorem of calculus, is a differentiable function of t, so

$$\xi'(t) = \frac{\xi(t+x_0) - \xi(t)}{\psi(x_0)} = \frac{\xi(x_0) - 1}{\psi(x_0)}\xi(t).$$

   In particular

$$\xi'(t) = s\xi(t)$$

   for a complex number s. By the theory of ODE, $\xi(t) = Ce^{st}$. Since $\xi(0) = 1$, we have $C = 1$. Moreover, $|\xi(t)| = 1$, so $e^{\Re(s)t} = 1$ for all real t, so $\Re(s) = 0$. Write $s = 2\pi ix$ for a real number x and we're done.

2. You can check the proof in the lecture notes, Theorem 7.1.1.