D-MATH
FS 2021
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# Solutions 5

Probabilistic Number Theory

**1** a. Let $U := \sum_{k\geq 2}(\zeta(k)-1)$; then

$$U = \sum_{j\geq 2}\sum_{k\geq 2}\frac{1}{j^k}.$$

We first try to find a closed form formula for the second sum. Let

$$T_j = \sum_{i\geq 2}\frac{1}{j^i}, \quad S_j = \sum_{i\geq 1}\frac{1}{j^i}.$$

Then

$$T_j = S_j - \frac{1}{j}, \quad T_j = \frac{S_j}{j}, \quad S_j = jT_j.$$

Now we compute the closed form

$$\begin{aligned} S_j &= T_j + \frac{1}{j} \\ S_j - \frac{S_j}{j} &= \frac{1}{j} \\ (j-1)S_j &= 1 \\ S_j &= \frac{1}{j} - 1 \\ T_j &= \frac{1}{j-1} - \frac{1}{j}. \end{aligned}$$

Let's write the sequence up

$$U = \Big(\frac{1}{1}-\frac{1}{2}\Big) + \Big(\frac{1}{2}-\frac{1}{3}\Big) + \Big(\frac{1}{3}-\frac{1}{4}\Big) + \dots$$

We would like to regroup, but in infinite sums it's not allowed when we are summing sequence of elements with alternating sign. Therefore we represent the sum as limit of finite sums sequence.

$$\begin{aligned} U &= \lim_{P\to\infty}\sum_{j=2}^{P}\Big(\frac{1}{j-1}-\frac{1}{j}\Big) \\ &= \lim_{P\to\infty}\Big(\frac{1}{1} + \Big(-\frac{1}{2}+\frac{1}{2}\Big) + \dots + \Big(-\frac{1}{P-1}+\frac{1}{P-1}\Big) - \frac{1}{P}\Big) \\ &= \lim_{P\to\infty}\Big(1-\frac{1}{P}\Big) = 1, \end{aligned}$$

So $U = 1$.

b. Let $U := \sum_{k\geq 1}(\zeta(k)-1)$. One has

$$V = \sum_{j\geq 2}\sum_{k\geq 1}\frac{1}{j^{2k}}.$$

By letting $M_j = \sum_{k\geq 1}\frac{1}{j^{2k}}$, we get

$$\begin{aligned} M_j &= \frac{1}{j^2} + \frac{M_j}{j^2} \\ j^2 M_j &= 1 + M_j \\ M_j &= \frac{1}{j^2-1} \\ &= \frac{2}{2(j^2-1)} \\ &= \frac{(j+1)-(j-1)}{2(j+1)(j-1)} \\ &= \frac{1}{2(j-1)} - \frac{1}{2(j+1)}. \end{aligned}$$

Let's compute $V$ as limit of finite sums sequence.

$$\begin{aligned} V &= \lim_{P\to\infty}\sum_{j=2}^{P}\Big(\frac{1}{2(j-1)} - \frac{1}{2(j+1)}\Big) \\ &= \lim_{P\to\infty}\frac{1}{2}\Big(1+\frac{1}{2}+\Big(-\frac{1}{3}+\frac{1}{3}\Big)+\cdots-\frac{1}{P}-\frac{1}{P+1}\Big) \\ &= \lim_{P\to\infty}\frac{1}{2}\Big(1+\frac{1}{2}-\frac{1}{P}-\frac{1}{P+1}\Big) = \frac{3}{4}. \end{aligned}$$

So $V = 3/4$.

**2** a. Write

$$\begin{aligned} \int_0^T |\zeta(\sigma+it)|^2 dt &= \int_0^T \sum_{n,m\geq 1}\frac{1}{n^{\sigma+it}m^{\sigma-it}}dt \\ &= \sum_{n,m\geq 1}\frac{1}{(nm)^\sigma}\int_0^T\left(\frac{n}{m}\right)^{-it}dt \\ &= \sum_{n\geq 1}\frac{1}{n^{2\sigma}}T + \sum_{n\neq m\geq 1}\frac{1}{(nm)^\sigma}\frac{e^{-it\log\left(\frac{n}{m}\right)}}{-i\log\left(\frac{n}{m}\right)}\Bigg|_0^T \\ &= T\zeta(2\sigma) + \sum_{n\neq m\geq 1}\frac{1}{(nm)^\sigma}\frac{e^{-it\log\left(\frac{n}{m}\right)}}{-i\log\left(\frac{n}{m}\right)}\Bigg|_0^T. \end{aligned}$$

Tyhe second summand is

$$\leq \sum_{n\neq m\geq 1}\frac{1}{(nm)^{\sigma}}\frac{2}{|\log\left(\frac{n}{m}\right)|}=4\sum_{n>m\geq 1}\frac{1}{(nm)^{\sigma}}\frac{1}{|\log\left(\frac{n}{m}\right)|}.$$

Note that if $|\log\left(\frac{n}{m}\right)|$ is small, the serie could not be convergent. We have

$$\log\left(\frac{n}{m}\right)=\log\left(1+\frac{n-m}{m}\right)\gg\begin{cases}\log 2 & \text{if } m<n/2\\ \frac{n-m}{m} & \text{if } m\geq n/2.\end{cases}$$

Therefore]

$$\sum_{n\neq m\geq 1}\frac{1}{(nm)^{\sigma}}\frac{e^{-it\log\left(\frac{n}{m}\right)}}{-i\log\left(\frac{n}{m}\right)}\Big|_0^T\leq 4\sum_{1\leq m<\frac{n}{2}}\frac{1}{(nm)^{\sigma}\log 2}$$
$$+4\sum_{\frac{n}{2}\leq m<n}\frac{1}{(nm)^{\sigma}}\frac{m}{n-m}=O(1).$$

b. By the Stirling's formula we have

$$\begin{aligned}|\zeta(1-\sigma-it)|&=2(2\pi)^{-\sigma}|\cos(\pi s/2)||\Gamma(s)||\zeta(\sigma+it)|\\ &\sim(2\pi)^{\frac{1}{2}-\sigma}|t|^{\sigma-\frac{1}{2}}|\zeta(\sigma+it)|,\end{aligned}$$

which implies $\mu(1-\sigma)=\sigma-\frac{1}{2}+\mu(\sigma)$. Since for $\sigma<0$ we have $1-\sigma>1$, one gets

$$\mu(\sigma)=\frac{1}{2}-\sigma$$

for $\sigma<0$. the inequality $\mu(\sigma)\leq-\frac{\sigma}{2}+\frac{1}{2}$, $0\leq\sigma\leq 1$ follows by convexity.

c. By the residue theorem,

$$\frac{1}{i}\int_{1/2-iT}^{1/2+iT}\zeta(s)ds=-2\pi\mathrm{Res}_{s=1}\zeta(s)$$
$$+\Big(\int_{2-iT}^{2+iT}+\int_{1/2-iT}^{2-iT}-\int_{1/2+iT}^{2+iT}\Big)\zeta(s)\frac{ds}{i}.$$

By the inequality for the Lindelöf function, the second and third integrals are bounded by $\ll T^{\frac{1}{4}+\varepsilon}$. The first one instead is

$$\begin{aligned}\int_{-T}^{T}\zeta(2+it)dt&=\sum_{n\geq 1}\int_{-T}^{T}\frac{1}{n^{2+it}}dt\\ &=2T+\sum_{n>1}\frac{n^{-iT}-n^{iT}}{n^2 i\log n},\end{aligned}$$

where the last summand converges.

3 Assume $\sum_{n\le x}\mu(n) = O(x^{\frac{1}{2}+\varepsilon})$; since for $\sigma > 1$ on has $\zeta(s)^{-1} = \sum_{n\ge 1}\frac{\mu(n)}{n^s}$, by partial summation of $\sum_{n\le x}\frac{\mu(n)}{n^s}$, and by letting $x\to +\infty$, we get

$$\frac{1}{\zeta(s)} = s\int_1^{+\infty} O\Big(\frac{1}{y^{\sigma+\frac{1}{2}-\varepsilon}}\Big)dy.$$

Now, the function $\frac{1}{y^{\sigma+\frac{1}{2}-\varepsilon}}$ is holomorphic if $\sigma+\frac{1}{2}-\varepsilon > 1 \iff \sigma > 1/2+\varepsilon\ \forall\varepsilon > 0$. This implies that $\zeta(s)$ has no zeros for $\sigma > 1/2$. But those zeros are symmetric with respect to the critical line, so this is the RH.

Conversely, by the Perron's approximation formula with $2 \le T \le x$, one has

$$\sum_{n\le x}\mu(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{\zeta(s)}\frac{x^s}{s}ds + O\Big(\frac{x\log x}{T}\Big), \quad c = 1+\frac{1}{\log x}.$$

Shift the integration line at $1/2+\varepsilon$; since we are assuming the RH, we'll not encounter any zero of $\zeta(s)$. By the estimate

$$\frac{1}{\zeta(s)} \ll (1+|t|^{\varepsilon}), \quad \sigma > \frac{1}{2}+\varepsilon,$$

valid under the RH, we have

$$\sum_{n\le x}\mu(n) \ll_\varepsilon x^{1/2+\varepsilon}T^{\varepsilon} + \frac{x}{T^{1-\varepsilon}}\log x.$$

Pick $T = x$ and the claim follows.

4 See https://blogs.ethz.ch/kowalski/2009/04/02/who-remembers-the-mills-number/