1.4.2021

<u>Recall</u> : explicit formula gives

$$\sum_{n \leq x} \Lambda(n)\chi(n) = \delta_q x - \sum_{\substack{L(\beta+i\gamma,\chi)=0 \\ 0 \leq \beta \leq 1 \\ |\gamma| \leq X}} \frac{x^{\beta+i\gamma}}{\beta+i\gamma} + (\text{error})$$

with $X = x^2$ [so error is really negligible]

so

$$|\psi(x;\chi) - \delta_q x| \lesssim x^{\max(\beta)} \sum_{\substack{\beta+i\gamma \\ |\gamma| \leq X}} \frac{1}{|\beta+i\gamma|}$$

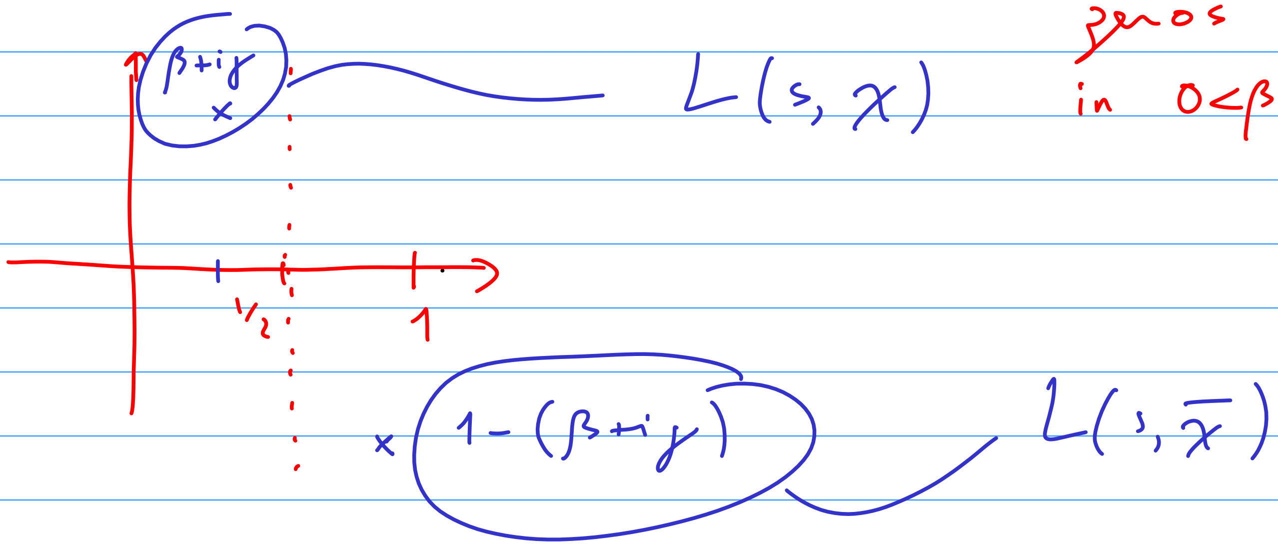
$$+ (\text{error})$$

<u>Two facts:</u> (essentially due to Riemann)

(1) if $\beta + i\gamma$ is a zero of $L(s,\chi)$ then $1-(\beta+i\gamma)$ is a zero of $L(s,\overline{\chi})$ ["functional equation"]

$\longrightarrow \max_{\chi} \max \beta \geq \frac{1}{2}$

[One can show: there are zeros in $0 < \beta \leq 1$]



so the best possible bound for x^{β} is $x^{\frac{1}{2}}$, if GRH(q) is true

(2) Need a bound on

$$\sum_{|\gamma| \leq X} \frac{1}{|\beta + i\gamma|}$$

$\Updownarrow$

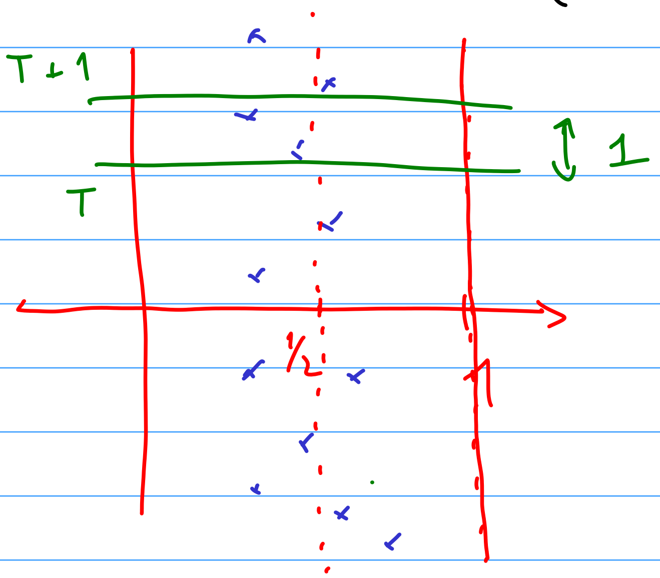
bound on number $m(T)$ of zeros with (say) $T \leq \gamma \leq T+1$

[because then

$$\sum_{|\gamma| \le X} \frac{1}{|\beta + i\gamma|} \le \sum_{-X \le k \le X} \frac{1}{k+1} \times m(k)$$

Here : one can show that

$$m(T) \le c \log(|T|+2)$$



$\longrightarrow$ this small strip contains $\ll \log(|T|+2)$ zeros, and

in fact

$$\left| \{ \beta + i\gamma \mid 0 < \beta \le 1, |\gamma| \le T \} \right|$$

$$= \frac{T}{2\pi} \log(qT/2\pi) - \frac{T}{\pi} + O(\log(qT))$$

[Prop. C.5.3]

so $m(T) \approx \log(T)$ on average over T.

So we get

$$\sum_{|\gamma| \leq x} \frac{1}{|\beta + i\gamma|} \leq c \sum_{|k| \leq X} \frac{1}{k+1} \log(|k|+2)$$

$$\leq C \log(X+2) \log(X)$$

(here C depends on q).

<u>Consequence</u>:

$$\psi(x;\chi) = \delta_q x + O\left(\sqrt{x} (\log x)^2\right)$$

(q fixed)

Further:

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O\left(\sqrt{x} (\log x)^2\right)$$

by orthogonality.

__Note:__ There are very strong reasons to believe in GRH(q).

3 _ Existence of the Rubinstein – Sarnak distribution

$q \geqslant 1$ fixed (ex. $q = 4$)

__Assume__ GRH(q)

$$\Omega_x = [1, x], \quad \mathbb{P}_x = \frac{1}{\log x} \frac{dt}{t}$$

$$\underline{N}_x : \Omega_x \longrightarrow C_{\mathbb{R}}\left((\mathbb{Z}/q\mathbb{Z})^{\times} \right)$$

is defined by

$$\boxed{\underline{N}_x (x)(a) = \frac{\log x}{\sqrt{x}} \left(\varphi(q)\, \pi(x; q, a) - \pi(x) \right)}$$

$\in \Omega_x$ $\quad (a, q) = 1$

(GRH(q))

Th. (Rubinstein-Sarnak)

$$\underline{N}_x \xrightarrow[x\to\infty]{\text{law}} N_q$$

for some r.v. N_q

1st Step:

Proposition (5.3.1)

For χ mod q let ψ_χ be the r.v. on Ω_x defined by

$$\psi_\chi(x) = \frac{1}{\sqrt{x}} \sum_{n \leq x} \Lambda(n)\, \chi(n).$$

related to zeros of L-functions

Then

$$\underline{N}_x(x)(a) = m_q(a) + \sum_{\chi \neq \varepsilon_q} \psi_\chi(x)\, \overline{\chi}(a) + O\left(\frac{1}{\log x}\right)$$

determistic

where $m_q(a) = -\sum_{\substack{\chi^2 = \varepsilon_q \\ \chi \neq \varepsilon_q}} \overline{\chi(a)}$.

Lemma. One has

$$m_q(a) = \begin{cases} 1 & \text{if } a \text{ not a square mod } q \\ 1 - \left|\{ b \bmod q \mid b^2 = 1 \}\right|. \end{cases}$$

Proof. We have

$$1 - m_q(a) = \sum_{\substack{\chi \bmod q \\ \chi^2 = \varepsilon_q}} \chi(a)$$

[because $(a,q)=1$, $\overline{\chi(a)} = \chi(a)$ if $\chi^2 = \varepsilon_q$]

$\chi^2 = \varepsilon_q$

$\Updownarrow$

$\chi = \tilde{\chi} \circ \pi$ where $\pi : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/q\mathbb{Z})^\times / (\text{squares})$

$$= \sum_{\substack{\chi \text{ char.} \\ \text{of } (\mathbb{Z}/q\mathbb{Z})^\times / (\text{squares})}} \chi(a \bmod \text{squares})$$

orthogonality for $(\mathbb{Z}/q\mathbb{Z})^\times / (\text{squares})$

$$= \begin{cases} 0 & a \text{ not a square} \\ \left|(\mathbb{Z}/q\mathbb{Z})^\times / (\text{squares})\right| \end{cases}$$

This is the statement because of

$$\left|(\mathbb{Z}/q\mathbb{Z})^\times / (\text{squares})\right| = \left|\operatorname{Ker}\left(\begin{array}{c} (\mathbb{Z}/q\mathbb{Z})^\times \circlearrowleft \\ x \longmapsto x^2 \end{array} \right)\right|$$

$$= \left|\{ b \mid b^2 = 1 \}\right|.$$

$\square$

<u>Ex.</u> $q = 4$

$$m_4(1) = 1 - 2 = -1$$

$$m_4(3) = 1$$

(More generally:

$$m_q(a) \leq 1$$

with equality $\Longleftrightarrow$ a is not a square.)

<u>N.B.</u> We will see that

$$m_q = \mathbb{E}(N_q)$$

so $m_4(1) < m_4(3)$ indicates a "bias" between

$$\underline{N}_{x,4}(3)$$

and $\underline{N}_{x,4}(1)$

(in favor of the first being larger) if $X \to +\infty$, meaning that $\pi(x;4,3) > \pi(x;4,1)$ "frequently".

Proof of Prop. -

def.

$$\underline{N}_x = \frac{\log x}{\sqrt{x}} \left(\varphi(q)\pi(x;q,a) - \pi(x) \right)$$

sum over $\chi \neq \varepsilon_q$

Orthogonality:

$$\varphi(q)\pi(x;q,a) = \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{p \leq x} \chi(p)$$

$$= \underbrace{\sum_{\substack{p \leq x \\ p \nmid q}} 1}_{\pi(x) + O(1)} + \sum_{\chi \bmod q}^{*} \overline{\chi(a)} \sum_{p \leq x} \chi(p)$$

q fixed; at most $\omega(q)$

(nb. of $p | q$

so

$$\underline{N}_\chi = \sum^*_{\chi \bmod q} \overline{\chi(a)} \boxed{\frac{\log x}{\sqrt{x}} \sum_{p \le x} \chi(p)}$$

$$+ O\left(\frac{\log x}{\sqrt{x}}\right)$$

negligible

Now let

$$\theta_\chi(x) = \frac{1}{\sqrt{x}} \sum_{p \le x} (\log p)\, \chi(p)$$

By definition of Λ:

$$\theta_\chi(x) - \psi_\chi(x) = -\frac{1}{\sqrt{x}} \sum_{\substack{p^k \le x \\ k \geq 2}}{}' \chi(p)^k \log p$$

$$= -\frac{1}{\sqrt{x}} \sum_{p \le \sqrt{x}} \chi(p)^2 \log p \; \underbrace{+ \; O\left(\frac{(\log x)^2}{x^{1/6}}\right)}_{\text{negligible}}$$

Two cases :

Case 1 : $\chi^2 \neq \varepsilon_q$ [recall $\chi \neq \varepsilon_q$]

$\rightsquigarrow$

$$\frac{1}{\sqrt{x}} \sum_{p \leq \sqrt{x}} (\chi^2)(p) \log p \ll \frac{1}{\sqrt{x}} x^{1/4} (\log x)^2$$

negligible

by (a form of) the GRH(q).

$$\left[\psi(\sqrt{x}\, ; \chi^2) \ll x^{1/4} (\log \sqrt{x})^2 \right]$$

Case 2 : $\chi^2 = \varepsilon_q$, $\chi \neq \varepsilon_q$

$$- \frac{1}{\sqrt{x}} \sum_{\substack{p \leq \sqrt{x} \\ p \nmid q}} \log p = \boxed{-1} + O\left(\frac{\log x}{x^{1/4}}\right)$$

deterministic, $\neq 0$

RH

Conclusion :

$$\boxed{\theta_\chi(x) - \psi_\chi(x) = - \delta_{\chi^2} + O\left(\frac{\log x}{x^{1/4}}\right)}$$

1 if $\chi^2 = \varepsilon_q$, 0 otherwise

Now we relate $\Theta_\chi(x)$ to

$$\frac{\log x}{\sqrt{x}} \sum_{p \le x} \chi(p).$$

Summation by parts:

$$\frac{1}{\sqrt{x}} \sum_{p \le x} \chi(p) = \frac{1}{\sqrt{x}} \sum_{p \le x} \chi(p) \log p \, \frac{1}{\log p}$$

$$\frac{1}{\log p} = g(p),$$

Compare with

$$\int_a^b f\, g = \left[F g\right]_a^b - \int_a^b F\, g'$$

where $F' = f$

$g(x) = \frac{1}{\log x}$ is smooth for $x \geq 2$

$$= g(x)\, \Theta_\chi(x) - g(2)\, \Theta_\chi(2)$$

$$- \frac{1}{\sqrt{x}} \int_2^x \sqrt{t}\; \Theta_\chi(t)\; g'(t)\, dt$$

$$= \frac{1}{\log x} \Theta_\chi(x) + \frac{1}{\sqrt{x}} \int_2^x \frac{\Theta_\chi(t)\, dt}{\sqrt{t}\,(\log t)^2}.$$

so

$$\frac{\log x}{\sqrt{x}} \sum_{p \le x} \chi(p) = \Theta_\chi(x) +$$

$$\frac{\log x}{\sqrt{x}} \int_2^x \frac{\theta_\chi(t)\,dt}{\sqrt{t}\,(\log t)^2}\,dt$$

One can show (see notes) easily that

$$\int_2^x \frac{\theta_\chi(t)\,dt}{\sqrt{t}\,(\log t)^2} = \int_2^x \frac{\psi_\chi(t)\,dt}{\sqrt{t}\,(\log t)^2} + O\left(\frac{\sqrt{x}}{(\log x)^3}\right)$$

We try again the obvious bound:

$$\left|\frac{\log x}{\sqrt{x}} \int_2^x \frac{\psi_\chi(t)\,dt}{\sqrt{t}\,(\log t)^2}\right| \ll \frac{\log x}{\sqrt{x}} \int_2^x \frac{dt}{\sqrt{t}}$$

GRH(q)

$$\approx \log x$$

$$\ll (\log t)^2$$

which is too big!

One can prove:

Cor C.5.11:

$$\int_2^x \psi(t;\chi)\,dt \ll x^{3/2}.$$

(See after vacation for explanation, coming from $\sum_{\substack{L(\beta+i\gamma,\chi) \\ = 0}} \frac{1}{|\beta+i\gamma|^2} < +\infty$)

Using integration by parts one deduces that

$$\frac{\log x}{\sqrt{x}} \int_2^x \frac{\psi_\chi(t)}{\sqrt{t}\,(\log t)^2}\,dt \ll \frac{1}{\log x}$$

which is good enough!

Conclusion:

$$N_x = \sum_{\chi\ (q)}^{\bullet} \overline{\chi(a)} \; \frac{\log x}{\sqrt{x}} \sum_{p \le x} \chi(p)$$

$+ \text{(error)}$

$$= \sum_{\chi(q)}^{\circ} \overline{\chi(a)} \; \theta_\chi(x) + (\text{error})$$

last step

$$= \sum_{\chi(q)}^{\circ} \overline{\chi(a)} \; \psi_\chi(x)$$

$$+ \sum_{\chi(q)}^{\circ} \bar{\chi}(a) \big(\theta_\chi(x) - \psi_\chi(x) \big)$$

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$$+ (\text{error})$$

$$= \sum_{\chi(q)}^{*} \overline{\chi(a)} \; \psi_\chi(x)$$

$$+ \sum_{\substack{\chi \neq \varepsilon_q \\ \chi^2 = \varepsilon_q}} - \overline{\chi(a)} + (\text{error})$$

$$= \boxed{m_q(a) + \sum_{\chi(q)}^{*} \overline{\chi(a)} \; \psi_\chi(x) + O\left(\frac{1}{\log x}\right)}$$

concluding the proof! $\square$