6.5.2021

Scheduling: no class on Thursday, May 13

Remark. Part (1) of Prop. 3.2.9 means precisely the following

(i) For a given $s \in U_\tau$, there exists $E_s \subset \Omega$ (underlying prob. space on which (X_p) is defined) with $\mathbb{P}(E_s) = 1$ s.t. for $\omega \in E_s$, we have

$$\prod_p (1 - X_p(\omega) p^{-s})^{-1} \qquad (*)$$

converges [in the sense that

$$\lim_{P \to +\infty} \prod_{p \leq P} (1 - X_p(\omega) p^{-s})^{-1}$$

exists]

(ii) For a given compact set $K \subset U_\tau$, there is a subset $F_K \subset \Omega$ s.t. $\mathbb{P}(F_K) = 1$ and for $\omega \in F_K$, the product (*) converges for all s ∈ K,

and
$$\begin{cases} s \longmapsto \prod_p \left(1 - X_p(\omega) p^{-s}\right)^{-1} \\ K \longrightarrow \mathbb{C} \end{cases}$$

belongs to $\mathcal{H}(K)$ (and defines an $\mathcal{H}(K)$-valued r.v.).

<u>Proof of Prop. 3.2.9.</u>

(1) Let $s \in U_\tau$, fixed.

For $P \geq 2$, we get

$$\left(\left| \frac{X_p}{p^s} \right| = \frac{1}{p^{\mathrm{Re}(s)}} < 1 \right)$$

$$\prod_{p \leq P} \left(1 - X_p p^{-s}\right)^{-1}$$

$$= \exp\left(\sum_{p \leq P}' \sum_{k \geq 1}' \frac{X_p^k}{p^{ks}} \right)$$

[Taylor expansion of $\log(1-z)$ for $|z| < 1$].

Write

$$\sum_{p \leq P} \sum_{k \geq 1} \frac{\chi_p^k}{p^{ks}} = D_{1,P}(s) + D_{2,P}(s)$$

where

$$D_{1,P}(s) = \sum_{p \leq P} \frac{\chi_p}{p^s}.$$

We see that the series

$$\sum_{p} \sum_{k \geq 2} \frac{\chi_p^k}{p^{ks}} \qquad (**)$$

converges absolutely because

$$\mathrm{Re}(ks) \geq 2\,\mathrm{Re}(s) > 2\tau > 1$$

(and $|\chi_p| = 1$), so

$$D_{2,P}(s) \longrightarrow (\text{this series } (**))$$

always.

__Claim.__ The series $\sum_p \frac{X_p}{p^s}$ converges almost surely.

(Indeed, since (X_p) are independent, we can use Kolmogorov's 3 series Theorem, and since $\left|\frac{X_p}{p^s}\right| \leq 1$, we need to check that:

$$\begin{cases} \sum_p \mathbb{E}\left(\frac{X_p}{p^s}\right) \quad \text{converges} \\ \sum_p \mathbb{V}\left(\frac{X_p}{p^s}\right) < +\infty \end{cases}$$

The first one is 0 $\left(\mathbb{E}(X_p) = 0\right)$, and the second is

$$\sum_p \frac{1}{p^{2Re(s)}} \mathbb{V}(X_p)$$

$$< \sum_p \frac{1}{p^{2\tau}} \cdot 1$$

$$< +\infty \;)$$

This together implies that

$$\prod_p \left(1 - \frac{X_p}{p^s}\right)^{-1}$$

converges almost surely.

Now we consider $K \subset U_2$ compact.

We use Lemma A.4.1:

<u>Lemma A.4.1</u>. Let $(a_n)_{n\geq 1}$ be a sequence of <u>complex numbers</u>, and let $s_0 \in \mathbb{C}$ be such that

$$\sum_{n\geq 1} a_n n^{-s_0}$$

converges.

Then the series

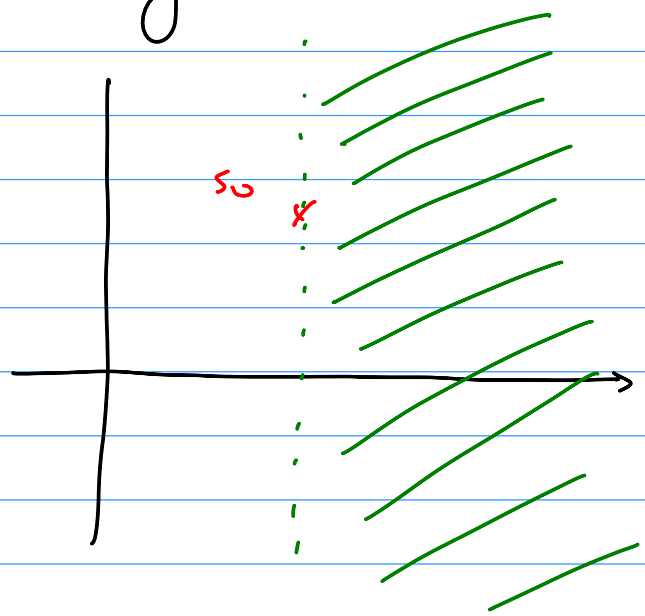
$$f(s) = \sum_{n\geq 1} a_n n^{-s}$$

converges for all $s \in \mathbb{C}$ such that

$$\mathrm{Re}(s) > \mathrm{Re}(s_0)$$

and does so uniformly on compact subsets. In particular, f is a holomorphic for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$.

N.B. Compare with power series and radius of convergence.

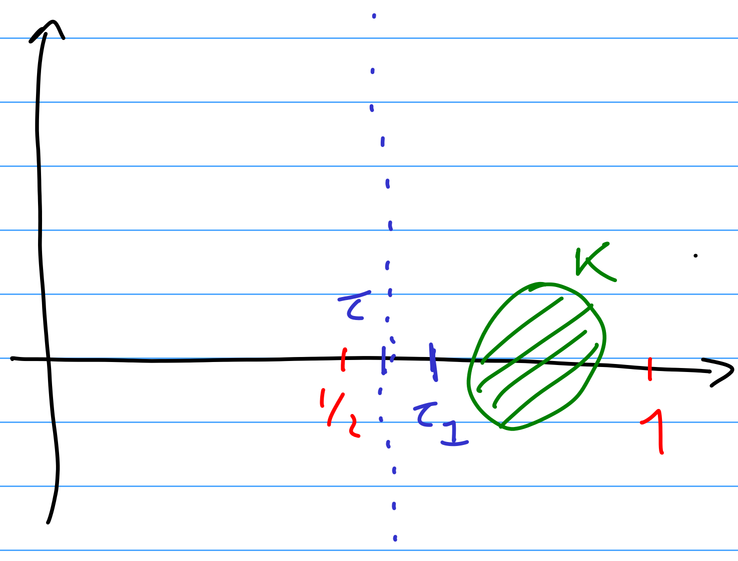


(Proof later)

Now pick τ_1 s.t.

$$\tau < \tau_1 < \mathrm{Re}(s)$$

for $s \in K$.



Since $\tau_1 \in U_\tau$, there exists $E_{\tau_1} \subset \Omega$ with $\mathbb{P}(E_{\tau_1}) = 1$

such that

$$\sum_p \frac{X_p(\omega)}{p^{\tau_1}}$$

converges for $\omega \in E_{\tau_1}$.

Pick $\omega \in E_{\tau_1}$; then Lemma A.4.1 [with $a_n = \begin{cases} X_p(\omega), & n = p \\ 0, & \text{otherwise} \end{cases}$

and $s_0 = \tau_1$]

shows that

$$f_\omega(s) = \sum_p \frac{X_p(\omega)}{p^s}$$

exists and is an element of $\mathcal{H}(K)$.

So we get (1) with "almost surely" meaning $\omega \in E_{\tau_1}$

(because the second part

$$\sum_{k \geq 2} \sum_p \frac{X_p^k}{p^{ks}}$$

converges for all ω).

Remark. This argument does not extend to $\mathrm{Re}(s) \geq \frac{1}{2}$ because

$$\sum_p \frac{X_p}{p^s}$$

does <u>not</u> converge almost surely if $\mathrm{Re}(s) = \frac{1}{2}$ (in fact diverges almost surely): indeed, the 3 Series Theorem is an "if and only if" and

$$\sum_p \mathbb{V}\left(\frac{X_p}{p^s}\right) = \sum_p \frac{1}{p^{2\mathrm{Re}(s)}}$$

$$= \sum_p \frac{1}{p} = +\infty.$$

(2) Let $s \in U_2$ be fixed. We consider the random series

$$\sum_{n \geq 1} \frac{X_n}{n^s}.$$

Difficulty: the summands X_n/n^s

are <u>not</u> independent, so we cannot use Kolmogorov's Th. (e.g. $X_4 = X_2^2$).

However we have:

<u>Claim.</u> The sequence $(X_n)_{n \geq 1}$ is <u>orthonormal</u>

i.e. for all $n, m \geq 1$

$$\mathbb{E}(X_n \overline{X_m}) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

(Indeed: if $n = m$, clear since $|X_n|^2 = 1$, and if $n \neq m$, there is a prime $p | n$ which has a different exponent for n and m, and then

$$\mathbb{E}(X_n \overline{X_m}) = \mathbb{E}(X_p^{v_n - v_m}) \times \mathbb{E}\left(X_{n/p^{v_n}} \overline{X_{m/p^{v_m}}}\right)$$

where

$$p^{v_n} \| n, \quad p^{v_m} \| m$$

and since $v_n - v_m \neq 0$, we get $\mathbb{E}(X_p^{v_n - v_m}) = 0$.)

[B.10.5]

Theorem (Menshov / Rademacher)

Let $(\Omega, \mathbb{P})$ be a probability space and (X_n) a sequence in $L^2(\Omega)$.
If (X_n) is orthonormal then for any sequence $(a_n)_{n\geq 1}$ of complex numbers, the condition

$$\sum_{n\geq 1} |a_n|^2 (\log n)^2 < +\infty$$

implies that the series

$$\sum_{n\geq 1} a_n X_n$$

converges almost surely.

Remark. (1) Let $\Omega = \mathbb{R}/\mathbb{Z}$, $\mathbb{P} = dx$ and

$$X_n(t) = e(nt) = e^{2i\pi n t}$$

(for $n \in \mathbb{Z}$). Then we get: if (a_n) satisfies

$$\sum_{n\in\mathbb{Z}} |a_n|^2 (\log n)^2 < +\infty,$$

then the Fourier series of the L^2-function $f(t) = \sum_{n\in\mathbb{Z}} a_n e(nt) \in L^2(\mathbb{R}/\mathbb{Z})$

converges for <u>almost all</u> $t \in \mathbb{R}/\mathbb{Z}$.

This was improved by <u>Carleson</u> to only require $\sum_{n\in\mathbb{Z}} |a_n|^2 < +\infty$

(but Kolmogorov proved that there exists $f \in L^1(\mathbb{R}/\mathbb{Z})$ such that the Fourier series $\sum_{n\in\mathbb{Z}} a_n e(nt)$ diverges for <u>all</u> t.

(2) Menshov also showed that the condition $\sum_{n\geq 1} |a_n|^2 (\log n)^2 < +\infty$ can <u>not</u> be improved (for general (X_n)).

Back to

$$\sum_{n\geq 1} \frac{X_n}{n^s} \qquad (s \in U_\tau \text{ fixed}).$$

We apply the Menshov/Rademacher Theorem to $(X_n)_{n\geq 1}$ and $a_n = \frac{1}{n^s}$. Then

$$\sum_{n\geq 1} |a_n|^2 (\log n)^2 = \sum_{n\geq 1} \frac{(\log n)^2}{n^{2Re(s)}}$$
$$\leq \sum_{n\geq 1} \frac{(\log n)^2}{n^{2\tau_1}}$$
$$< +\infty \quad \text{since } \tau_1 > \frac{1}{2}$$

(Again $Re\, s = \frac{1}{2}$ is out of reach).

Now by the same method as in (1), we deduce that for τ_1 s.t. $\tau < \tau_1 < Re(s)$ for $s \in K$, the function

$$g_\omega(s) = \sum_{n\geq 1} \frac{X_n(\omega)}{n^s}$$

$(s \in K)$

exists and belongs to $\mathcal{H}(K)$ if $\omega \in F_{\tau_1}$, where $\mathbb{P}(F_{\tau_1}) = 1$ and $\sum_{n\geq 1} \frac{X_n(\omega)}{n^{\tau_1}}$ exists for $\omega \in F_{\tau_1}$.

Now for the last step (showing that the product and the series coincide almost surely), we use analytic continuation.

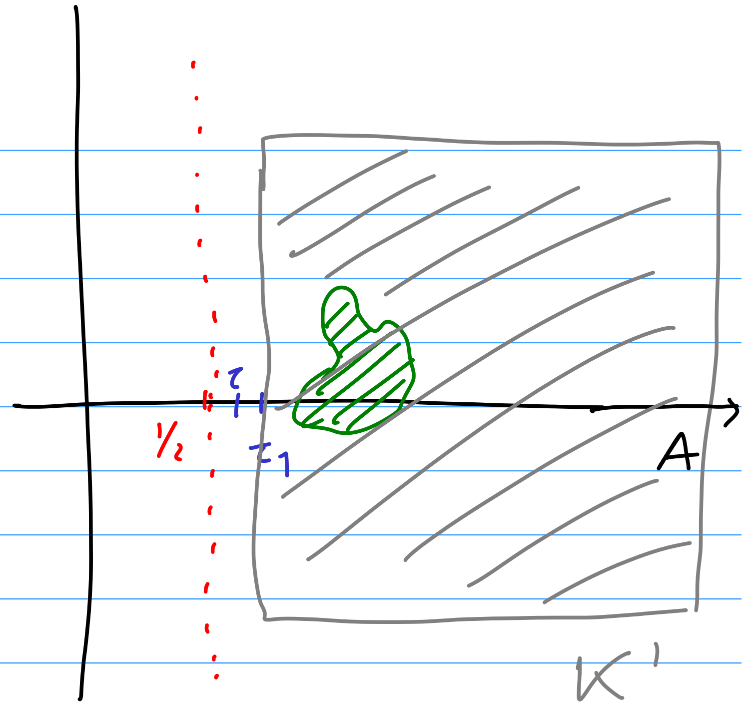
Let $A \geq 2$ s.t.

$$K \subset \underbrace{\left\{ \tau_1 \leq \mathrm{Re}(s) \leq A, \ |\mathrm{Im}(s)| \leq A \right\}}_{K'} \subset U_\tau$$

(for $\tau_1 > \tau$).

$K' \subset U_\tau$

is compact



Let $F \subset \Omega$ be s.t. $\mathbb{P}(F) = 1$

and
$$\left\{ \begin{array}{l} \displaystyle\sum_{n \geq 1} \frac{X_n(\omega)}{n^s} \\ \\ \displaystyle\prod_p \left(1 - X_p(\omega) p^{-s}\right)^{-1} \end{array} \right.$$

belong to $\mathcal{H}(K')$ for $\omega \in F$

[given by previous argument].

For $\omega \in F$ and $s \in K'$ s.t.

$$\mathrm{Re}(s) \geq \frac{3}{2}$$

we have

$$\sum_{n \geq 1} \left| \frac{X_n(\omega)}{n^s} \right| < +\infty$$

and $X_n(\omega)$ is a multiplicative function of n, so that

$$\sum_{n \geq 1} \frac{X_n(\omega)}{n^s} = \prod_p \left(1 + \frac{X_p(\omega)}{p^s} + \frac{X_{p^2}(\omega)}{p^{2s}} + \cdots \right)$$

$$= \prod_p \frac{1}{1 - \frac{X_p(\omega)}{p^s}}$$

[Lemma C.1.4].

This means that the series and the product (for this ω) coincide for such s, hence they coincide on all of K' by analytic continuation (and continuity), hence also on $K \subset K'$.

This ends the proof of Prop. 3.2.9. $\square$

Sketch of proof of Lemma A.4.1

$(a_n)_{n \geq 1}$ complex numbers

s_0 s.t. $\displaystyle\sum_{n \geq 1} \frac{a_n}{n^{s_0}}$ converges

Replacing a_n by $a_n n^{-s_0}$ we can assume that $s_0 = 0$, so

$$\sum_{n \geq 1} a_n \quad \text{converges.}$$

Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\sum_{n \geq 1} a_n n^{-s} \quad ?$$

Let $s_{N,M} = a_N + \dots + a_M$ for $M \geq N \geq 1$.

Assumption $\Leftrightarrow$ $\displaystyle\lim_{\substack{M \geq N \geq 1 \\ \to +\infty}} s_{N,M} = 0$.

We need to study

$$\sum_{n=N}^{M} a_n n^{-s}$$

for $M \geq N \longrightarrow \infty$.

Use summation by parts:

$$\sum_{n=N}^{M} a_n n^{-s} = \frac{a_M}{M^s} - \sum_{N \le n < M} s_{N,n} \left(\frac{1}{(n+1)^s} - \frac{1}{n^s} \right)$$

so

$$\left| \sum_{n=N}^{M} a_n n^{-s} \right| \le \frac{|a_M|}{M^\sigma} + \sum_{N \le n < M} |s_{N,n}| \underbrace{\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right|}$$

$(\sigma = Re(s) > 0)$

Now observe

$$\frac{1}{(n+1)^s} - \frac{1}{n^s} = s \int_n^{n+1} x^{-s-1} \, dx$$

so

$$\left| \underline{\qquad\qquad\qquad} \right| \le |s| \int_n^{n+1} x^{-\sigma-1} \, dx$$

$$= \frac{|s|}{\sigma} \left(\frac{1}{(n+1)^\sigma} - \frac{1}{n^\sigma} \right)$$

$\Longrightarrow$

$$\left| \sum_{n=N}^{M} a_n n^{-s} \right| \le \frac{|a_M|}{M^\sigma} + \frac{|s|}{\sigma} \max_{N \le n \le M} |s_{N,n}|$$

$$\sum_n \left(\frac{1}{(n+1)^\sigma} - \frac{1}{n^\sigma} \right)$$

$$\leq \frac{|s|}{\sigma} \underbrace{\left(\max_{N \leq n \leq M} |s_{N,n}| \right)}_{\to 0 \text{ by Cauchy's Criterion}} \underbrace{\left(\frac{1}{N^\sigma} - \frac{1}{M^\sigma} \right)}_{\to 0 \text{ for } M \geq N \to \infty}$$

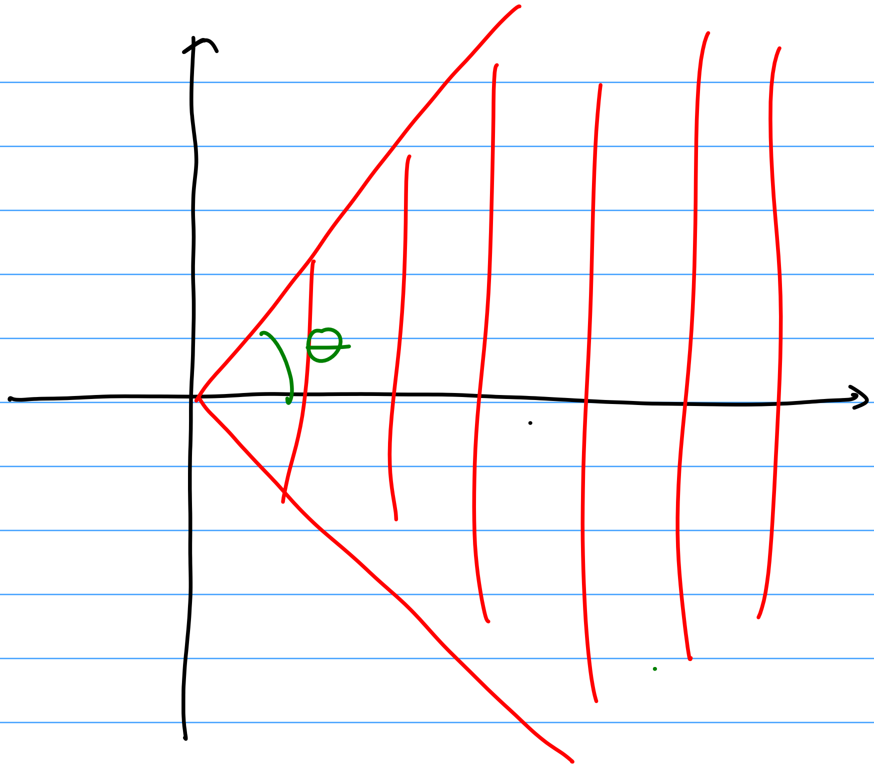
lack of uniformity!

so $\sum_{n \geq 1} a_n n^{-s}$ converges by Cauchy's Criterion.

Moreover we get uniform convergence in the regions of the form

$$Re(s) > 0, \quad \frac{|s|}{Re(s)} \leq A$$

for some $A > 0$.



$\theta \longrightarrow \frac{\pi}{2}$ as $A \longrightarrow +\infty$

This contains any given compact set if A is large enough!

$\square$