10.5.2021

Scheduling:

No class on 13.5

Class on 17.5 (Monday)

Next exercise class on 20.5

4. Step 2: a formula in the critical strip

Goal: we have a function $f(s)$, holomorphic (or meromorphic) in $\mathrm{Re}(s) > 1$, with a convergent series expansion, and with analytic continuation to $\mathrm{Re}(s) > 0$; for $s_0 \in \mathbb{C}$ with

$$0 < \mathrm{Re}(s_0) < 1,$$

how can we compute $f(s_0)$?

(**Ex.** $f(s) = \zeta(s)$

or $f(s) = \sum_{n \geq 1} \frac{X_n(\omega)}{n^s}$)

The following proposition shows how this may be done.

Proposition (A. 4. 3)

Let $0 < \sigma_0 < 1$

$(a_n)_{n \geq 1}$ in $\mathbb{C}$, $|a_n| \leq 1$

$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$

so f is holomorphic for $Re(s) > 1$.

Assume:

(1) f extends to a meromorphic function for $Re(s) > \sigma_0$, with at most a simple pole at $s = 1$ with residue $c \in \mathbb{C}$.

missing in notes

(2) f has at most polynomial growth in a region

$$\sigma_1 \leq \mathrm{Re}(s) \leq A$$

for fixed $0 < \sigma_0 < \sigma_1 < 1$

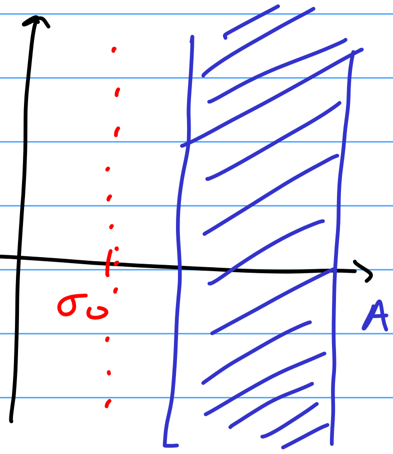
and $A \geq 1$

i.e. $\exists\, C, B \geq 0$ s.t.

$$|f(s)| \leq C\,(1 + |s|)^B$$

or $|\mathrm{Im}(s)|$

for s in this region



Let $s = \sigma + it$ with $0 < \sigma_0 < \sigma \leq 1$.

Then ("smoothing formula") for $N \geq 1$

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}\,\varphi\!\left(\frac{n}{N}\right)$$

finite sum

$$- c\, N^{1-s}\,\hat{\varphi}(1-s)$$

(0 if f has no pole at $s=1$)

$$- \frac{1}{2i\pi} \int_{(-\delta)} f(s+w)\, N^w\, \hat{\varphi}(w)\, dw$$

$\mathrm{Re}(w) = -\delta$

$$|N^w| = N^{-\delta}$$

where (a) $\varphi : [0, +\infty[\longrightarrow [0,1]$ is smooth and compactly supported (so $\varphi(x) = 0$ for x large enough)

and $\varphi(0) = 1$, and

$$\hat{\varphi}(s) = \int_0^{+\infty} \varphi(x)\, x^s \, \frac{dx}{x}$$

is the Mellin transform of φ, which is meromorphic for $Re(s) > -1$ with a simple pole at $s = 0$ with residue $\varphi(0) = 1$.

(b) $\delta > 0$ is such that

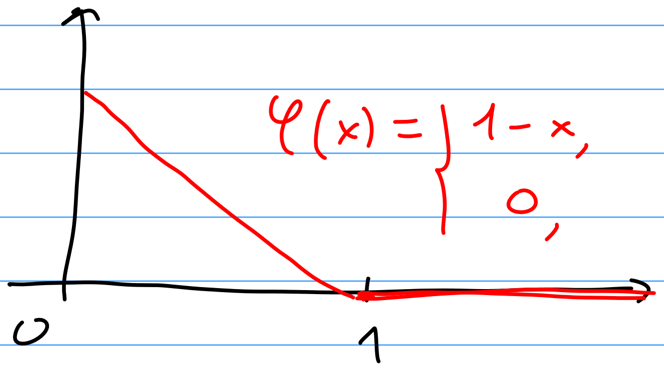
$$\sigma - \delta > \sigma_0 .$$

Note: the key point is the <u>smoothing effect</u> of the $\varphi\left(\frac{n}{N}\right)$.

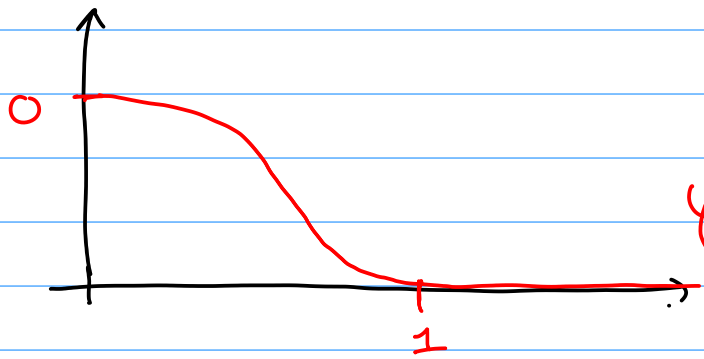
Variants include:

$$\sum_{n \leq N} a_n \left(1 - \frac{n}{N} \right)$$

(Cesaro means / Féjer kernel)
for Fourier series



$$\varphi(x) = \begin{cases} 1-x, & 0 \leq x \leq 1 \\ 0, & x \geqslant 1 \end{cases}$$



$\varphi \in C_c^\infty([0,+\infty[)$

One knows that functions φ as in the Prop. do exist.

(A.3.1)
<u>Lemma.</u> Let φ be as above.

(1) $\hat{\varphi}$ is merom. for $\mathrm{Re}(s) > -1$ with pole at $s=0$ with residue $\varphi(0)$.

(2) $\hat{\varphi}$ decays fast in strips:

$\forall \delta > 0,$

$\forall k \geq 1, \quad \exists C_k$ s.t.

$$|\hat{\varphi}(s)| \leq \frac{C_k}{(1+|s|^k)}$$

for $\begin{cases} -1+\delta \leq \mathrm{Re}(s) \leq 3 \\ |\mathrm{Im}(s)| \geq 1 \end{cases}$

Proof. (Harmonic analysis: Fourier transform exchanges decay properties and smoothness properties).

(1)

$$\hat{\varphi}(s) = \int_0^{+\infty} \varphi(x)\, x^s \frac{dx}{x} \qquad (\mathrm{Re}(s)>0)$$

$$= \left[\frac{1}{s} x^s \varphi(x)\right]_0^{\infty} - \frac{1}{s}\int_0^{\infty} \varphi'(x)\, x^s\, dx$$

$$= 0 - 0 - \frac{1}{s}\int_0^{\infty} \varphi'(x)\, x^s\, dx$$

so $\quad s\,\hat{\varphi}(s) = -\int_0^{\infty} \varphi'(x)\, x^s\, dx \quad (*)$

holomorphic for

$\mathrm{Re}(s) > -1$

so $\hat{\varphi}$ is meromorphic for $\mathrm{Re}(s) > -1$ with at most a pole at $s = 0$; and

$$\lim_{s \to 0} s\,\hat{\varphi}(s) = -\int_0^{+\infty} \varphi'(x)\,dx$$

$$= \varphi(0) \neq 0$$

so $\hat{\varphi}$ has a simple pole with residue $\varphi(0)$ at $s = 0$.

(2) We repeat integrating by parts k times

$$\hat{\varphi}(s) = \frac{(-1)^k}{s(s+1)\cdots(s+k-1)} \int_0^{+\infty} \varphi^{(k)}(x)\, x^{s+k-1}\,dx$$

(generalizes $(*)$)

$$\Rightarrow \quad |\hat{\varphi}(s)| \ll \frac{C_k}{(1+|s|)^k}$$

for $\begin{cases} |\operatorname{Im}(s)| \geq 1 \\ -1+\delta \leq \operatorname{Re}(s) \leq 3 \end{cases}$

by simple computations.

$\square$

<u>Proof of the smoothing formula</u> :

$$\sum_{n \geq 1} \frac{a_n}{n^s} \varphi\left(\frac{n}{N}\right)$$

$$\overset{\text{Mellin inversion}}{=} \sum_{n \geq 1} \frac{a_n}{n^s} \times \frac{1}{2i\pi} \int_{(2)} \hat{\varphi}(w) \left(\frac{n}{N}\right)^{-w} dw$$

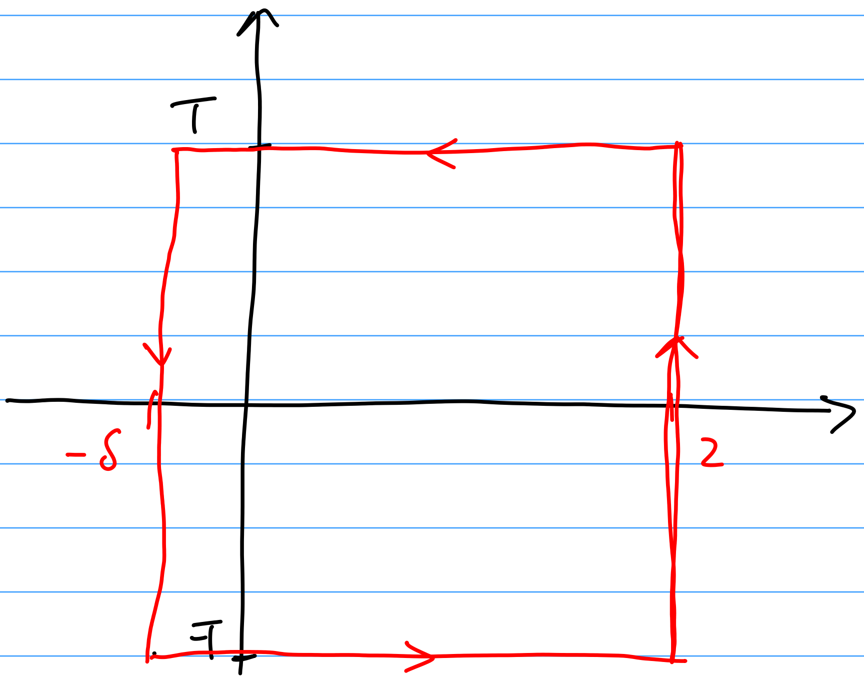
integrate over the line $\operatorname{Re}(w) = 2$, oriented $\uparrow$

$$= \frac{1}{2i\pi} \int_{(2)} \hat{\varphi}(w) N^{+w} f(s+w)\, dw$$

(version of Plancherel formula).

Let $T \geq 1$ be a parameter.

Consider the path R_T, and



$g(w) = \hat{\varphi}(w) N^w f(s+w)$ on R_T and the interior.

g is meromorphic on a small open neighborhood of R_T (and its interior), if $\begin{cases} \sigma - \delta > \sigma_0 \\ -\delta > -1 \end{cases}$ (since

$$\mathrm{Re}(s+w) \geq \sigma - \delta > \sigma_0),$$

so we can apply Cauchy's Theorem and get

Key Residue!

$$\frac{1}{2i\pi} \int g(w)\, dw = 1 \times f(s)$$

$$+ \; c\, \hat{\varphi}(1-s)\, N^{1-s}$$

since :

(i) $\hat{\varphi}$ has a pole at $w = 0$ with residue $\varphi(0) = 1$ [and $s \neq 1$]

(ii) $f(s+w)$ has (maybe) a pole at $w = 1-s$ with residue c

Now we let $T \longrightarrow +\infty$ in the integral. Then

$$\frac{1}{2i\pi} \int_{2-iT}^{2+iT} g(w)\, dw \xrightarrow[T\to\infty]{} \frac{1}{2i\pi} \int_{(2)} g(w)\, dw$$

(because the RHS exists)

$$= \sum_{n\geq 1} \frac{a_n}{n^s} \varphi\left(\frac{n}{N}\right)$$

(orientation)

$$\frac{1}{2i\pi} \int_{-\delta+iT}^{-\delta-iT} g(w)\, dw \xrightarrow[T\to\infty]{} - \frac{1}{2i\pi} \int_{(-\delta)} g(w)\, dw$$

so if the horizontal parts tend to 0, we get

$$\sum_{n \geq 1} \frac{a_n}{n^s} \varphi\left(\frac{n}{N}\right)$$

$$= \frac{1}{2i\pi} \int_{(-\delta)} g(w)\, dw$$

$$+ f(s)$$

$$+ c\, N^{1-s} \hat{\varphi}(1-s),$$

finishing the proof.

But:

$$\left| \frac{1}{2i\pi} \int_{-\delta+iT}^{2+iT} g(w)\, dw \right| \leq \frac{1}{2\pi} \int_{-\delta}^{2} \left| \hat{\varphi}(x+iT) \right| N^{x} \left| f(s+x+iT) \right| dx$$

$-\delta + iT \longleftarrow 2+iT$

in a fixed strip

$$\ll N^2 \int_{-\delta}^{2} (1+T)^{-k} \times (1+T)^{A}\, dx$$

for arbitrary k [$\hat{\varphi}$ decays fast]
and some A [pol. growth of f]

$\rightsquigarrow$ contribution is

$$\ll \frac{N^2}{T} \xrightarrow{T \to \infty} 0$$

by taking $k = A + 1$.

$\square$

<u>Note:</u> this is often done without particular justifications in analytic number theory.

5_ <u>Key estimate for Bagchi's Theorem</u>

<u>Proposition</u> (3.2.11, 3.2.12)

$D = \{ s \in \mathbb{C} \mid |s - \frac{3}{4}| < r \}$
$(r < 1/4)$
Let φ be as above.

(X_p) i.i.d on $\mathbb{S}'$, defined on some $(\Omega, \mathbb{P})$

(1) Let

$$Z = \sum_{n \geq 1} \frac{X_n}{n^s}$$

as $\mathcal{H}(D)$-valued r.v. and

$$Z_N = \sum_{n \geq 1} \frac{X_n}{n^s} \varphi\left(\frac{n}{N}\right)$$

(also $\mathcal{H}(D)$-valued).

There exists $\delta > 0$ s.t.

$$\mathbb{E}\left(\| Z - Z_N \|_\infty\right) \ll N^{-\delta}$$

$\int_\Omega (\dots)\, d\mathbb{P}$ $\qquad$ $\sup_{s \in \overline{D}} |\dots|$

(2) Let $T \geq 1$, $\Omega_T = [-T, T]$,

$$\underline{Z}_T(t) = \left(s \longmapsto \zeta(s+it)\right)$$

and

$$\underline{Z}_{T,N}(t) = \left(s \longmapsto \sum_{n \geq 1} \varphi\left(\frac{n}{N}\right) n^{-s-it}\right)$$

Then there exists $\delta > 0$ s.t.

$$\mathbb{E}_T\left(\| \underline{Z}_T - \underline{Z}_{T,N} \|_\infty\right) \ll N^{-\delta} + \frac{N}{T}.$$

$$\frac{1}{2T}\int_{-T}^{T}(\ldots\ldots)\,dt$$

Assuming this, we prove Bagchi's Theorem as follows.

It suffices to prove that for any $f: \mathcal{H}(D) \longrightarrow \mathbb{C}$ Lipschitz - continuous and bounded, we have

$$\mathbb{E}_T\left(f\left(\underline{z}_T\right)\right) \xrightarrow[T\to\infty]{} \mathbb{E}\left(f(z)\right).$$

Let $C \geqslant 0$ be a Lipschitz constant.

Let $N \geqslant 1$ be a parameter.

Then

$$\mathbb{E}_T\left(f\left(\underline{z}_T\right)\right) - \mathbb{E}\left(f\left(z\right)\right)$$

$$= \mathbb{E}_T\Big(f(\underline{z}_T) - f(\underline{z}_{T,N})\Big)$$
$$+ \mathbb{E}_T\big(f(\underline{z}_{T,N})\big) - \mathbb{E}\big(f(z_N)\big)$$
$$+ \mathbb{E}\big(f(z_N) - f(z)\big).$$

want to $\longrightarrow 0$ as $T \to \infty$

so

$$|\text{---}\ \text{-}\ \text{---}| \leq \mathbb{E}_T\Big(\big|f(\underline{z}_T) - f(\underline{z}_{T,N})\big|\Big)$$
$$+ \Big|\mathbb{E}_T\big(f(\underline{z}_{T,N})\big) - \mathbb{E}\big(f(z_N)\big)\Big|$$
$$+ \mathbb{E}\Big(\big|f(z_N) - f(z)\big|\Big)$$

f Lipschitz

$$\leq C\, \mathbb{E}_T\big(\|\underline{z}_T - \underline{z}_{T,N}\|_\infty\big)$$
$$+ C\, \mathbb{E}\big(\|z_N - z\|_\infty\big)$$
$$+ \Big|\mathbb{E}_T\big(f(\underline{z}_{T,N})\big) - \mathbb{E}\big(f(z_N)\big)\Big|.$$

$$\ll N^{-\delta} + \frac{N}{T}$$

$$+ \left| \mathbb{E}_T\left(f\left(\underline{z}_{T,N}\right)\right) - \mathbb{E}\left(f\left(z_N\right)\right)\right|.$$

Pick $\varepsilon > 0$, and fix N s.t.

$$N^{-\delta} < \varepsilon.$$

Observe:

$$\underline{z}_{T,N}(s) = \sum_{n \geq 1} \varphi\left(\frac{n}{N}\right) n^{-s-it}$$

$$z_N(s) = \sum_{n \geq 1} \varphi\left(\frac{n}{N}\right) n^{-s} X_n$$

are "the same" continuous functions of $(p^{-it})_p$ and $(X_p)_p$.

because finite sums!

We know $(p^{-it}) \xrightarrow{\text{law}} (X_p)$

so by composition

$$\underline{Z}_{T,N} \xrightarrow[T \to \infty]{\text{law}} Z_N$$

so

$$\mathbb{E}_T\left(f\left(\underline{Z}_{T,N}\right)\right) \xrightarrow[T \to \infty]{} \mathbb{E}\left(f\left(Z_N\right)\right)$$

So the difference we wanted to bound will be $\leq 4\varepsilon$ for all T large enough.

This concludes the proof.