12.4.2021

Recall: why do we have (under GRH(q))

$$\int_2^x \Big(\sum_{n \le t} \Lambda(n) \chi(n) \Big) \, dt \ll x^{3/2}$$

(without extra logarithm)?

Use the explicit formula to get that this is:

$$\int_2^x \Bigg(\sum_{\substack{L(\frac{1}{2}+i\gamma,\chi)=0 \\ |\gamma| \le x}} \frac{t^{1/2+i\gamma}}{\frac{1}{2}+i\gamma} + \text{error} \Bigg) dt$$

$$= \sum_{\substack{|\gamma| \le x \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{1}{1/2+i\gamma} \, \frac{x^{3/2+i\gamma} - 2^{3/2+i\gamma}}{\frac{3}{2}+i\gamma}$$

$$+ (\text{error})$$

so the modulus is at most

$$\ll \sum_{\substack{|\gamma| \le x \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{x^{3/2}}{|\frac{1}{2}+i\gamma| \, |\frac{3}{2}+i\gamma|}$$

$$\ll x^{3/2} \sum_{|\gamma| \leq x} \frac{1}{|\frac{1}{2}+i\gamma|^2}$$

But one knows that

$$\sum_{L(\frac{1}{2}+i\gamma,\, \chi)=0} \frac{1}{|\frac{1}{2}+i\gamma|^2}$$

is convergent [check by splitting for $|\gamma| \leq x$ into intervals of length 1, in which there are $\ll (\log x)$ zeros, so we compare the sum with

$$\sum_{m \geq 1} \frac{1}{m^2} \log(m) < +\infty \Big]$$

[Key fact: not too many zeros of $L(s,\chi)$; compare with problems about Laplace eigenvalues in dim ≥ 2]

So we have proved

$$\underline{N}_X(a) = \underbrace{\boxed{m_q(a) + \sum_{\chi \neq \varepsilon_q} \bar{\chi}(a)\, \varphi_\chi}}_{\underline{M}_X(a)} + \left(\text{something converging to } 0 \text{ in probab.}\right)$$

It is elementary (probabilistically) that $\underline{N}_X$ converges in law

$$\Updownarrow$$

$\underline{M}_X$ ______________

(with the same limit).

We use the explicit formula [with param. X] again to write

normalization of φ_χ

$$\underline{M}_X(a) = m_q(a) + \sum_{\chi \neq \varepsilon_q} \bar{\chi}(a) \left[\sum_{\substack{L(\frac{1}{2}+i\gamma,\chi)=0 \\ |\mathrm{Im}\,\gamma| \leq X}} \frac{x^{1/2+i\gamma-1/2}}{\frac{1}{2}+i\gamma} + O\left(\frac{(\log X)^2}{X^{1/2}}\right)\right]$$

and again the convergence in law is equivalent to that of

$$\boxed{m_q(a) + \sum_{\chi \neq \varepsilon_q} \overline{\chi(a)} \sum_{\substack{L(\frac{1}{2}+i\gamma, \chi)=0 \\ |\gamma| \leq X}} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}}$$

oscillates

Explicit formula:

$$\frac{1}{\sqrt{x}} \sum_{n \leq x} \Lambda(n) \chi(n) = - \sum_{\rho} \frac{x^{\rho - \frac{1}{2}}}{\rho} + O\left(\frac{x^{1/2} (\log x)^2}{X} \right)$$

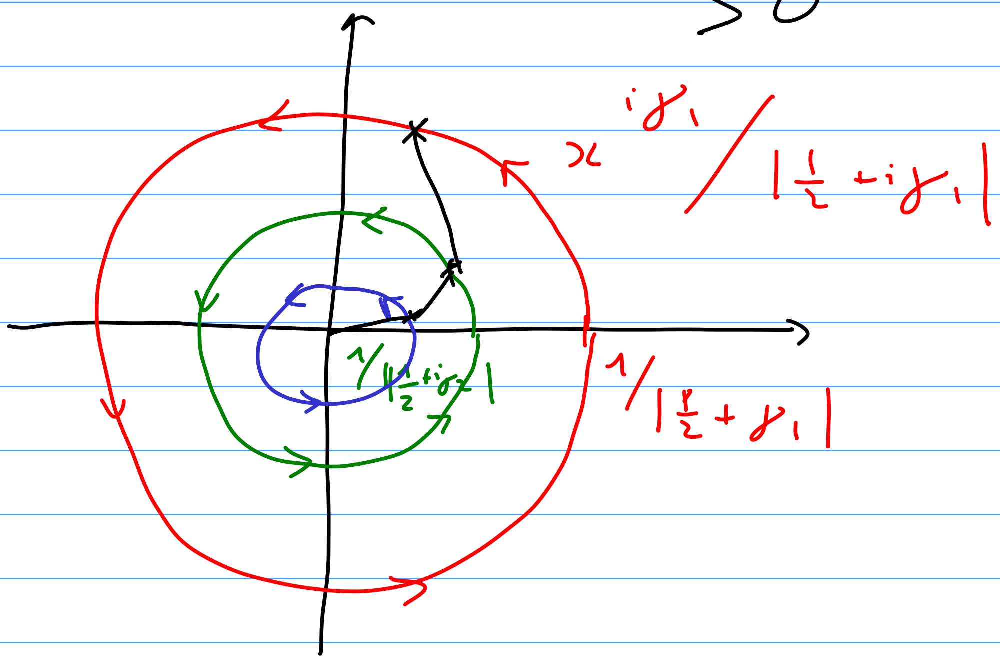
so if $x \leq X$ then error is

$$\leq \frac{(\log X)^2}{X^{1/2}}$$

So in other words we have the following kind of random quantities:

$$\sum_{\substack{\text{finite sum} \\ \text{of real} \\ \text{numbers } \gamma}} \underbrace{\alpha(\gamma)}_{>0} \, x^{i\gamma}$$

$$\boxed{x^{i\gamma} = e^{i\gamma \log x}}$$



We can expect some complicated behavior arising from this!

We have a basic theorem:

Th. (5.3.3) Let $k \geq 1$.

Let $F \subset \mathbb{R}$ be finite of size k, let $\alpha(\gamma)$ for $\gamma \in F$ be elements of $\mathbb{C}^k$.

Then the random vectors

$$\begin{cases} \Omega_X \longrightarrow \mathbb{C}^k \\ x \longmapsto \sum_{\gamma \in F} x^{i\gamma} \alpha(\gamma) \end{cases}$$

converge in law

(and we will see how to compute the limit in principle).

Proof - Note that this vector is $f(W_X)$ where

$$W_X : \Omega_X \longrightarrow (\mathbb{S}^1)^F$$
$$x \longmapsto (x^{i\gamma})_{\gamma \in F}$$

and

$$f((z_\gamma)_{\gamma\in F}) = \sum z_\gamma \, \alpha(\gamma)$$

is a continuous map $(\mathbb{S}^1)^F \longrightarrow \mathbb{C}^k$

so it suffices to prove that

$$W_X \xrightarrow{\text{law}} W \quad [\text{for some } W]$$

and then

$$f(W_X) \xrightarrow{\text{law}} f(W).$$

Key fact is : $(\mathbb{S}^1)^F$ is a compact group (with multipli-
-cation]

The behavior of W_X follows from :

<u>Theorem</u> ("Kronecker's Theorem")

Let $d \geqslant 1$, $z = (z_i)_{1 \leq i \leq d} \in \mathbb{R}^d$

T = closure of $\{ y z \mid y \in \mathbb{R} \}$ in $(\mathbb{R}/\mathbb{Z})^d$

For $B > 0$, let μ_B be the image of $\frac{1}{B}\, dt$ on $[0, B]$ under the map

$$t \longmapsto [tz] \in (\mathbb{R}/\mathbb{Z})^d$$

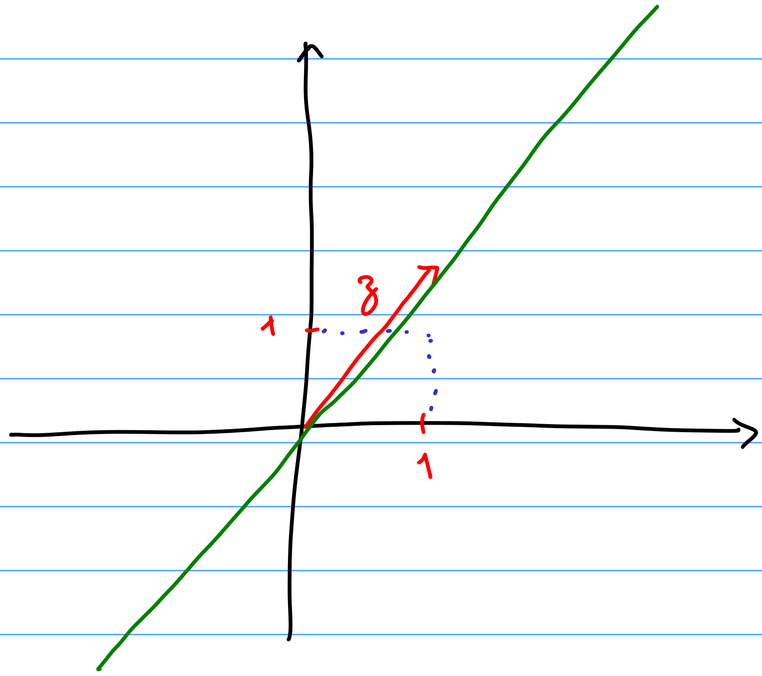
[so

$$\mu_B(A) = \frac{1}{B} \operatorname{Leb}(\{ t \in [0, B] \,/\, tz \bmod \mathbb{Z}^d \in A \})]$$

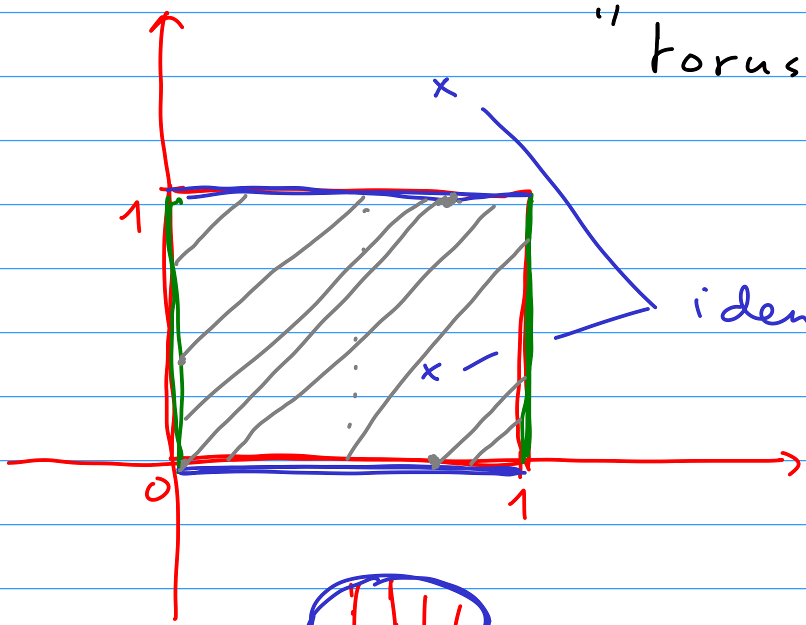
Then

$$\mu_B \xrightarrow[B \to +\infty]{\text{law}} \text{“uniform measure on } T\text{”}$$

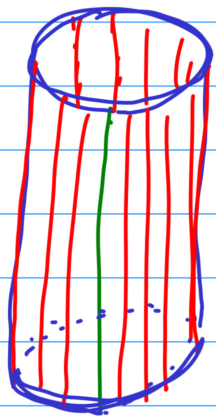
Ex. $d = 2$, $z = (1, \sqrt{2}) \in \mathbb{R}^2$

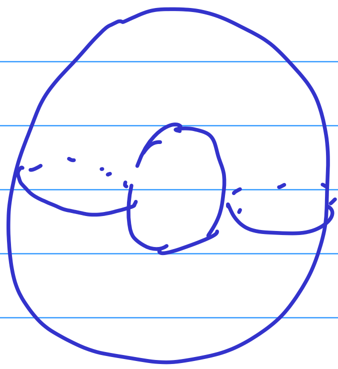


T is the set of points on the line $\mathbb{R}z$ projected mod $\mathbb{Z}^2$ to the "torus" $(\mathbb{R}/\mathbb{Z})^2$



identified in $(\mathbb{R}/\mathbb{Z})^2$





In that case, we will see that

$$T = (\mathbb{R}/\mathbb{Z})^2$$

and then the line $\mathbb{R}(1,\sqrt{2}) \bmod \mathbb{Z}^2$ is "uniformly distributed", which means like the Lebesgue measure.

More generally, the notion of a "uniform" random variable makes sense for any compact group.

<u>Theorem</u> [Haar]

(1) If G is a <u>locally compact</u> group (<u>ex.</u> $\mathbb{R}^d$, $(\mathbb{R}/\mathbb{Z})^d$, $SL_2(\mathbb{R})$) then there exist Radon measures ≥ 0 (Haar measure) μ_G on G which are <u>left-invariant</u>:

$$\int_G f(g)\, d\mu_G(g) = \int_G f(xg)\, d\mu_G(g)$$

for all x. This is unique up

to multiplication by $\alpha > 0$.

(2) If G is compact then there is a unique <u>probability</u> Haar measure; we will denote it μ_G; it is then automatically also right invariant.

If a G-valued r.v. Y has law μ_G, we say that Y is uniformly distributed on G.

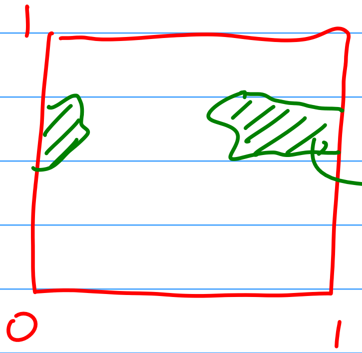
<u>Ex.</u> (1) $T = \overline{\{y\xi \mid y \in \mathbb{R}\}}$ is a subgroup of $(\mathbb{R}/\mathbb{Z})^d$, compact, so has a probability Haar measure μ_T and in Kronecker's Theorem, this is the limit.

(2) On $\mathbb{R}$: Lebesgue measure is a Haar measure

(3) On $\mathbb{R}/\mathbb{Z}$: image of Lebesgue on $[0,1]$ is the canonical Haar measure,

usually denoted dt

(4) On $(\mathbb{R}/\mathbb{Z})^d$: $dt_1 \cdots dt_d$ is the prob. Haar measure



Lebesgue of the green set in $[0,1]^2$ is the Haar measure of the corresponding subset of $(\mathbb{R}/\mathbb{Z})^2$.

(5) If G is finite, then

$$\mu_G(\{a\}) = \frac{1}{|G|}$$

is the probability Haar measure.

Back to Kronecker's Theorem.

We use the following criterion:

Weyl Criterion

Let G be a compact abelian group. Let (μ_n) be a sequence of Radon probability measures on G.

Then

$$\mu_n \xrightarrow{\text{law}} \mu_G$$

$$\Updownarrow$$

$\forall\ \chi : G \longrightarrow S^1 \subset \mathbb{C}^\times$ continuous homomorphism (= "character"), $\chi \neq 1$

$$\int_G \chi(g)\, d\mu_n(g) \xrightarrow[n\to\infty]{} 0.$$

Remarks:

(1) since $|\chi| = 1$, continuous, we have $\chi \in L^1(G, \mu_n)$.

(2) the RHS of the limit

is $$0 = \int_G \chi(g)\, d\mu_G(g)$$

because $\chi \neq 1$

More generally: for χ_1, χ_2 characters of G we have

$$\boxed{\int_G \chi_1(g)\overline{\chi_2(g)}\, d\mu_G = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2 \\ 1 & \text{if } \chi_1 = \chi_2 \end{cases}}$$

("orthogonality relations")

[Indeed: if $\chi_1 = \chi_2$ then we get $$\int_G |\chi_1|^2 = \int_G 1 = 1;$$

if $\chi_1 \neq \chi_2$, pick $y \in G$ with $$\chi_1 \overline{\chi_2}(y) \neq 1$$

$\overline{\chi_2}$ " χ_2^{-1}

Then

$$\int_G \chi_1(g)\overline{\chi_2(g)}\, d\mu_G = \int_G \chi_1(yg)\overline{\chi_2(yg)}$$

Haar

$$= \underbrace{(\chi_1 \overline{\chi_2})(y)}_{\neq 1} \int_G \chi_1(g) \overline{\chi_2(g)}$$

χ_1, χ_2 homomorphisms

$$\Longrightarrow \int_G \chi_1(g) \overline{\chi_2(g)} \, d\mu_G(g) = 0$$

Ex. $G = (\mathbb{R}/\mathbb{Z})^d$, $d \geq 1$

One can show that the characters of G are given by

$$\chi(t_1, \dots, t_d) = e^{2i\pi(t_1 h_1 + \dots + t_d h_d)}$$

for some unique $h = (h_i) \in \mathbb{Z}^d$

(well-defined because $e^{2i\pi h} = 1$ if $h \in \mathbb{Z}$).

In particular for $d = 1$:

$$\chi(t) = e^{2i\pi h t}$$

for some $h \in \mathbb{Z}$.

$$\boxed{\text{Notation: } e(z) = e^{2i\pi z} \text{ for } z \in \mathbb{C}.}$$