15.4.2021

Remark. Take $G = \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$

$(\mu_G = dt)$ $t \longmapsto e^{it}$

The characters are

$$\chi_h : t \longmapsto e(ht) = e^{2i\pi ht}, \quad h \in \mathbb{Z}$$

Suppose $\mu_n = f_n(t)\, dt$, $n \geq 1$

where f_n is 1-periodic and continuous

Then

$$\int_G \chi_h(t)\, d\mu_n(t) = \int_G f_n(t)\, e^{2i\pi ht}\, dt = (-h)\text{-th Fourier coefficient of the function } f_n$$

So the condition in Weyl Criterion is that all Fourier coefficients of

f_n converge to 0 as $n \to \infty$, $h \neq 0$.

One deduces then for any convergent subsequence $(\mu_{n_k})_{k \geq 1}$, the limit μ satisfies

$$\forall\, h \neq 0, \quad \int \chi_h \, d\mu = 0 .$$

Again if we knew that $\mu = f(t)\,dt$, then f has all $\neq 0$ Fourier coefficients equal to 0.

$\Rightarrow$ f is constant $\Rightarrow$ $f = 1$ since μ is a probability measure

$\Rightarrow$ $\mu = dt$ is the Haar measure.

<u>In general</u>: the characters of a compact group form an orthonormal basis of $L^2(G, \mu_G)$.

Proof of Kronecker's Theorem

(Why should we expect it to be true? μ_B are measures on $(\mathbb{R}/\mathbb{Z})^d$ supported on T, so (the) limit is also supported on T; could it be converging to μ_T?)

We apply the <u>Weyl Criterion</u>: let $\chi : T \longrightarrow S^1$, $\chi \neq 1$, be a character of T. We have

$$\int_T \chi(x)\, d\mu_B(x)$$

$$= \frac{1}{B} \int_0^B \chi(y\gamma)\, dy.$$

$\in \mathbb{R}^d$

Fact: any character χ of $T \subset (\mathbb{R}/\mathbb{Z})^d$ extends to a character $\tilde{\chi}$ of $(\mathbb{R}/\mathbb{Z})^d$.

This means that there exists a

$$h = (h_1, \dots, h_d) \in \mathbb{Z}^d$$

such that $\underset{\underset{T}{\cap}}{\chi(x)} = e(\underbrace{x \cdot h}_{\sum_{i=1}^{d} x_i h_i})$, and

so

$$\int_T \chi(x)\, d\mu_B(x) = \frac{1}{B}\int_0^B e^{2i\pi y \overbrace{(\mathfrak{z}\cdot h)}^{\text{real number}}}\, dy$$

<u>Claim:</u> because χ is not equal to 1 on T, we have $\mathfrak{z}\cdot h \neq 0$ (otherwise $\chi(y\mathfrak{z}) = e^{2i\pi y(\mathfrak{z}\cdot h)} = 1$ for all $y \in \mathbb{R}$, so $\chi = 1$ on T by definition of T.)

So the integral is

$$\frac{1}{B}\ \underbrace{\frac{1}{2i\pi(\mathfrak{z}\cdot h)}}_{\text{fixed}}\ \underbrace{\left(e^{2i\pi B(\mathfrak{z}\cdot h)} - 1\right)}_{|\cdot| \leq 2}$$

$$\xrightarrow[B\to\infty]{} 0.$$

This concludes the proof.

$\square$

In practice, we often use the case $T = (\mathbb{R}/\mathbb{Z})^d$ [i.e. the image mod $\mathbb{Z}^d$ of the line $\mathbb{R}z$ is <u>dense</u> in $(\mathbb{R}/\mathbb{Z})^d$].

<u>Lemma.</u> We have $T = (\mathbb{R}/\mathbb{Z})^d$ $\iff$ $(z_1, \ldots, z_d)$ are <u>$\mathbb{Q}$-linearly independent</u>.

(<u>Ex.</u> <u>$d = 2$</u> : $z = (1, \sqrt{2})$

$\longrightarrow T = (\mathbb{R}/\mathbb{Z})^2$

<u>$d = 3$</u> : $z = (1, \sqrt{2}, \sqrt{3})$

$\longrightarrow T = (\mathbb{R}/\mathbb{Z})^3$)

<u>Proof.</u> <u>$\Rightarrow$</u> : by contraposition, if we have

$$n_1 z_1 + \dots + n_d z_d = 0$$

with $n_i \in \mathbb{Q}$, not all 0, then multiply by a suitable $N \geq 1$ to get a relation with $n_i \in \mathbb{Z}$, not all 0; then T is contained in

makes sense in $(\mathbb{R}/\mathbb{Z})^d$

$$T' = \{ (x_1, \dots, x_d) \in (\mathbb{R}/\mathbb{Z})^d \ / \ n_1 x_1 + \dots + n_d x_d = 0 \}.$$

(T' is closed)

and $T' \neq (\mathbb{R}/\mathbb{Z})^d$ (because $\underset{(n_i)}{n} \neq 0$).

$\Leftarrow$: trick: the assumption implies that

$$\frac{1}{B} \int_0^B e(y\,(h \cdot z))\,dy \xrightarrow[B \to \infty]{} 0$$

$$= \frac{1}{B} \frac{e(B(h \cdot z)) - 1}{2i\pi\,(h \cdot z)}$$

for all $h \in \mathbb{Z}^d - \{0\}$ [because $h \cdot z = h_1 z_1 + \dots + h_d z_d \neq 0$ by linear independance], so by the Weyl Criterion, we then get

$$\mu_B \longrightarrow \mu_{(\mathbb{R}/\mathbb{Z})^d} \quad \text{automatically,}$$

and this means that $T = (\mathbb{R}/\mathbb{Z})^d$.

$\square$

Proof of Th. 5.3.3

Recall: $F \subset \mathbb{R}$ finite

$$\alpha(\gamma) \in \mathbb{C}^k \quad \text{for } \gamma \in F$$

Goal: $V_X \begin{cases} \Omega_X \longrightarrow \mathbb{C}^k \\ x \longmapsto \sum_{\gamma \in F} \alpha(\gamma) x^{i\gamma} \end{cases}$

converges in law.

Let $f : \begin{cases} (\mathbb{S}^1)^F \longrightarrow \mathbb{C} \\ (x_\gamma) \longmapsto \sum_{\gamma \in F} \alpha(\gamma) x_\gamma \end{cases}$

continuous; then

$$V_X = f(W_X)$$

where

$$\omega_X \begin{cases} \Omega_X \longrightarrow (\mathbb{S}^1)^F \\ x \longmapsto (x^{i\gamma})_{\gamma \in F} \end{cases}$$

Note that there is an isomorphism

$$\begin{cases} (\mathbb{R}/\mathbb{Z})^F \xrightarrow{e} (\mathbb{S}^1)^F \\ (z_\gamma) \longmapsto (e(z_\gamma))_{\gamma \in F} \end{cases}$$

[both of groups and homeomorphism]

For $x \in \Omega_X$, we have

$$x^{i\gamma} = e^{2i\pi\gamma \frac{\log x}{2\pi}}$$

$$= e(z_\gamma \log x)$$

where $z_\gamma = \frac{\gamma}{2\pi}$.

Moreover for $x \in \Omega_X$, $\log x \in \underbrace{[0, \log X]}$

and "the corresponding measure on $[0, \log(X)]$ is uniform":

$$\mathbb{E}_X(f(x)) = \frac{1}{\log X} \int_1^X f(x) \frac{dx}{x}$$

$$= \frac{1}{\log X} \int_0^{\log X} f(e^t)\, dt .$$

$t = \log(x)$

$dt = \frac{dx}{x}$

So for $f : (\mathbb{S}^1)^F \longrightarrow \mathbb{C}$, continuous

$$\mathbb{E}_X (f(W_X))$$

$$= \frac{1}{\log X} \int_0^{\log X} f(e(y\mathfrak{z}))\, dy$$

where $\mathfrak{z} = (\mathfrak{z}_\gamma) = \left(\frac{\gamma}{2\pi}\right)_{\gamma \in F}$, i.e.

$$= \int_{\mathbb{T}} f(e(w))\, d\mu_{\log X}(w)$$

(with notation from Kronecker's Theorem). Since $\mu_{\log X}$ converges in law $\underbrace{}_{\text{as } X \to +\infty}$ (Kronecker's Th.), so does W_X, and the limit is e(uniform measure on $\mathbb{T}$), which is the uniform measure on the closed subgroup $e(T) \subset e((\mathbb{R}/\mathbb{Z})^F)$

$$= (\mathbb{S}^1)^F .$$

[And if γ is $\mathbb{Q}$-linearly independent then $e(T) = (\mathbb{S}^1)^F$].

$\square$

<u>Summary</u>: for $\sum_{\gamma \in F} \alpha(\gamma) x^{i\gamma}$ on Ω_X, we have the limit in law

$$\sum_{\gamma \in F} \alpha(\gamma) x^{i\gamma} \xrightarrow[X \to \infty]{\text{law}} \sum_{\gamma \in F} \alpha(\gamma) X_\gamma$$

$$(x \in \Omega_X)$$

where $(X_\gamma)_{\gamma \in F}$ is $(\mathbb{S}^1)^F$-valued and is uniformly distributed on the closed subgroup $\boxed{T_F = \{ (x^{i\gamma})_{\gamma \in F} \mid x > 0 \}}$

If the $\gamma \in F$ are $\mathbb{Q}$-linearly independent, then $(X_\gamma)_{\gamma \in F}$ is a vector of <u>independent</u> r.v. on $\mathbb{S}^1$, all uniformly distributed, and the limit is

$$\sum_{\gamma \in F} \alpha(\gamma) X_\gamma$$

is sum of <u>independent</u> r.v.

<u>Remark</u>: if we take

$$\tilde{\Omega}_x = [1, x+1]$$

$$\tilde{\mathbb{P}}_x = \frac{1}{x} dt \text{ on } \tilde{\Omega}_x$$

<u>then</u> for $F = \{\gamma\}$, the ω_x <u>does not converge in law</u>:

$$\omega_x(t) = t^{i\gamma} \text{ on } \tilde{\Omega}_x$$

and for $f = \mathrm{Id} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, we get

$$\tilde{\mathbb{E}}_x(f(\omega_x)) = \frac{1}{x} \int_1^{x+1} t^{i\gamma} dt$$

$$= \frac{1}{X} \frac{1}{1+i\gamma} \left(X^{1+i\gamma} - 1 \right)$$

$$= \frac{X^{i\gamma}}{1+i\gamma} + \underbrace{o(1)}_{\text{as } X \to +\infty}$$

has $\underline{\text{no limit}}$ as $X \to \infty$, in fact has the circle of radius $\frac{1}{|1+i\gamma|}$ as set of limit points

This explains why one uses Ω_X and $\mathbb{P}_X$ to study the Chebychev bias!

Back to the Rubinstein-Sarnak measure

We have to study

$$\underline{M}_X(a) = m_q(a) - \sum_{\chi \neq \varepsilon_q} \overline{\chi(a)} \sum_{\substack{L(\frac{1}{2}+i\gamma,\chi)=0 \\ |\gamma| \leq X}} \frac{x^{i\gamma}}{1/2+i\gamma}$$

as $X \longrightarrow +\infty$.

This is not quite literally the situation of Th. 5.3.3, because the set of γ's varies with X; instead of being a fixed F.

But Th. 5.3.3 says that if we fix $1 \leq T$ then for $X \to +\infty$, the r. v.

$$m_q(a) - \sum_{\chi \neq \varepsilon_q} \overline{\chi(a)} \sum_{|\gamma| \leq T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}$$

finite, since $L(s,\chi)$ are holomorphic

has a limit as $X \longrightarrow \infty$ $(x \in \Omega_X)$.

This is the right setup for Prop. B.4.4!