17.5.2021

Scheduling:

20.5 : Exercise (last one!)

24.5 : Holiday

Last three lectures: class

Remark. The proof of Bagchi's Th. is similar to Prop. B.4.4; however we could not apply Prop. B.4.4 because that result depends on Riesz Representation Th. for probability measure, which only applies to finite-dimensional spaces (unlike $\mathcal{H}(D)$).

Proof of Proposition 3.2.11

We consider (1), since (2) is similar (a bit more complicated).

Goal:

$$\mathbb{E}\big(\underbrace{\|z - z_N\|_\infty}_{\sup_{s\in\overline{D}} |z(s)-z_N(s)|}\big) \ll \,?$$

where

$$z = \sum_{n\geq 1} \frac{X_n}{n^s}$$

and

$$z_N = \sum_{n\geq 1} \frac{X_n}{n^s}\,\varphi\left(\frac{n}{N}\right).$$

Lemma. (Polynomial growth)

(1) Let s_0 have $\frac{1}{2} < \mathrm{Re}(s) < 1$, $\omega \in \Omega$ s.t. $\sum_{n\geq 1} X_n(\omega)\, n^{-s_0}$ converges. Then for $\delta > 0$

$$\left|\sum_{n\geq 1} \frac{X_n(\omega)}{n^s}\right| \ll 1 + |s|$$

depends on ω

for $\mathrm{Re}(s) \geq \mathrm{Re}(s_0) + \delta$.

(2) Let $\delta > 0$. Then for

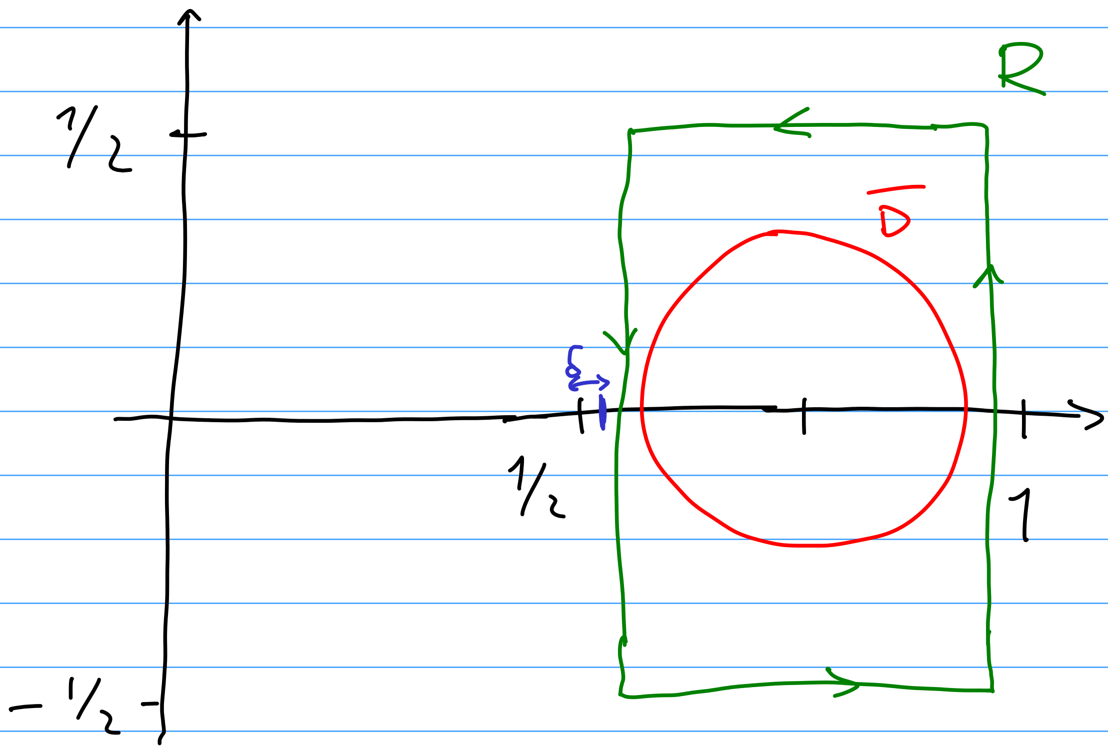
$$\mathrm{Re}(s) \geq \frac{1}{2} + \delta$$

we have

$$\mathbb{E}\left(|Z(s)| \right) \ll 1 + |s|.$$

Assume that this lemma holds.

Setup:



Idea: we will use contour integrals along $R = [a, b] \times \left[-\frac{1}{2}, \frac{1}{2} \right]$

which is chosen so that

$$\boxed{\frac{1}{2} < a < b < 1}$$

and $\overline{D} \subset \mathring{R}$. We need to use the smoothing formula for $s \in R$. So pick some $\delta > 0$ so that

$$\boxed{\frac{1}{2} + \delta < a}$$

Note that $Z(s)$ and $Z_N(s)$ make sense as $\mathcal{H}(R)$ - valued r.v, and their restrictions to $\overline{D}$ give the "original" Z and Z_N.

Now we apply first smoothing: almost surely, we have

$\mathrm{Re}(-) = \mathrm{Re}(s) - \delta$

$$Z(s) - Z_N(s) = -\frac{1}{2i\pi} \int_{(-\delta)} Z(s+w)\, \hat{\varphi}(w)\, N^w \, dw$$

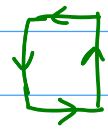
for all $s \in R$.

Mer. for $\mathrm{Re}(w) > -1$

[Note: $w \mapsto Z(s+w)$ is a.s. holomorphic for $\mathrm{Re}(w) = -\delta$]

For $v \in \overline{D}$, we apply Cauchy's Theorem along R to get

$$Z(v) - Z_N(v) = \frac{1}{2i\pi} \int_{\partial R} \left(Z(s) - Z_N(s)\right) \frac{ds}{s-v}$$



where we note that $\boxed{|s-v| \gg 1}$ for all s, v (by the choice of R)

$\Rightarrow$

$$Z(v) - Z_N(v) = \frac{1}{2i\pi} \int_{\partial R} \frac{ds}{s-v} \left(- \frac{1}{2i\pi} \int_{(-\delta)} Z(s+w)\, \hat{\varphi}(w)\, N^{w}\, dw \right)$$

so

$$\| Z - Z_N \|_\infty \ll N^{-\delta} \int_{\partial R} \int_{\mathbb{R}} | Z(s+iu-\delta) |\, | \hat{\varphi}(-\delta+iu) |\, du\, |ds|$$

Take the expectation to get:

$$\mathbb{E}\left(\|Z - Z_N\|_\infty\right)$$

$$\ll N^{-\delta} \int_{\partial R} \int_{\mathbb{R}} \mathbb{E}\left(\overbrace{|Z(s+iu-\delta)|}^{\text{Re} \geq \frac{1}{2}+\gamma \text{ for some } \gamma>0}\right) |\hat{\varphi}(-\delta+iu)|\, du\, |ds|$$

$$\ll N^{-\delta} \sup_{s\in\partial R} \int_{\mathbb{R}} \mathbb{E}\left(|Z(s+iu-\delta)|\right) |\hat{\varphi}(-\delta+iu)|\, du$$

Recall:

1. Lemma $\Rightarrow \mathbb{E}\left(|Z(s+iu-\delta)|\right) \ll 1+|u|$
2. $\varphi \in C^\infty([0,+\infty[)$, compact support $\Rightarrow \hat{\varphi}(-\delta+iu) \ll \dfrac{1}{(1+|u|)^{3}}$

any exponent can be used

so the integral is $\ll 1$ for all $s \in \partial R$.

This gives the bound we want.

Proof of the Lemma

(1) see A.4.2 for the details.

(2) Pick σ_0 such that

$$\frac{1}{2} < \sigma_0 < \frac{1}{2} + \delta .$$

We know that

$$\sum_{n \geq 1} \frac{X_n}{n^{\sigma_0}}$$

converges almost surely, say for $\omega \in A \subset \Omega$, with $\mathbb{P}(A) = 1$.

In particular, for $\omega \in A$, the partial sums

$$S_u(\omega) = \sum_{n \leq u} \frac{X_n(\omega)}{n^{\sigma_0}}$$

are bounded for $u \geq 1$.

Now let s have $\mathrm{Re}(s) \geq \frac{1}{2} + \delta$.

For $\omega \in A$, $Z_\omega(s)$ converges

and

$$Z_\omega(s) = \sum_{n \geq 1} \frac{X_n(\omega)}{n^s}$$

$$= \lim_{x \to \infty} \left(\sum_{n \leq x} \frac{X_n(\omega)}{n^s} \right)$$

$$\frac{X_n(\omega)}{n^s} = a_n f(n)$$

where

$$a_n = \frac{X_n(\omega)}{n^{\sigma_0}}$$

$$f'(x) = (\sigma_0 - s)\, x^{-s+\sigma_0-1} \longleftarrow f(x) = x^{-s+\sigma_0}$$

summation by parts

$$= \lim_{x \to \infty} \left(S_x(\omega) f(x) + (s-\sigma_0) \int_1^x S_u(\omega)\, u^{\sigma_0 - 1 - s}\, du \right)$$

$$= 0 + (s - \sigma_0) \int_1^\infty \underbrace{S_u(\omega)\, u^{\sigma_0 - 1 - s}}_{\ll u^{-1-\mathrm{Re}(s)+\sigma_0}}\, du$$

since $S_x(\omega) \ll 1$

and $\mathrm{Re}(s - \sigma_0) = \mathrm{Re}(s) - \sigma_0 > 0$

Now take the expectation

$$\mathbb{E}(|Z(s)|) \ll (1 + |s|) \int_1^\infty \mathbb{E}(|S_u|)\, u^{\sigma_0 - 1 - \mathrm{Re}(s)}\, du.$$

To bound $\mathbb{E}(|S_u|)$, we use the Cauchy-Schwarz inequality:

$$\mathbb{E}(|S_u|) \leq \mathbb{E}(|S_u|^2)^{1/2}$$

$$= \mathbb{E}\left(\left| \sum_{n \leq u} \frac{X_n}{n^{\sigma_0}} \right|^2 \right)^{\frac{1}{2}}$$

the (X_n) are orthonormal

$$= \sum_{n \leq u} \frac{1}{n^{\sigma_0}} \ll 1$$

since $2\sigma_0 > 1$.

So

$$\mathbb{E}(|Z(s)|) \ll (1+|s|) \underbrace{\int_1^\infty u^{\sigma_0 - 1 - \mathrm{Re}(s)} \, du}_{\text{convergent for } \mathrm{Re}(s) \geq \frac{1}{2}+\delta > \sigma_0}$$

as claimed.

$\square$

This concludes the proof of 3.2.11.

For 3.2.12 ($\mathbb{E}_T(\|Z_T - Z_{T,N}\|_\infty)$)

the arguments follow the same lines. One needs to know similar polynomial growth statements:

(1) for $\mathrm{Re}(s) \geq \frac{1}{2} + \delta$, $\delta > 0$,

and $|\mathrm{Im}(s)| \geq 1$, we have

$$|\zeta(s)| \ll 1 + |s|$$

(in fact much better is known).

(2) $$\frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma_0 + it)| \, dt \ll 1$$

for fixed $\frac{1}{2} < \sigma_0 < 1$.

There are references in the notes for these.

5. Final remarks

(1) To compute $\mathrm{Supp}(Z(s)) \subset \mathcal{H}(D)$, one needs a lot of complex analysis, and the Prime Number Theorem.

(2) Selberg's Theorem involves much more number theory:

- $(p^{-it}) \xrightarrow{\text{law}} (X_p)$ has to be made quantitative
- ideas from sieve theory
- asymptotic formula for

$$\frac{1}{2T}\int_{-T}^{T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 \left(\frac{n}{m}\right)^{it} dt$$

$$[\text{for } n, m \geq 1]$$

uniformly / quantitatively.

Selberg

(3) There are fascinating

conjectures about mysterious links between $\zeta(\frac{1}{2}+it)$ and Random Unitary Matrices (e.g "Keating - Snaith conjecture")

Chapter V

[Ch. 6 in the notes]

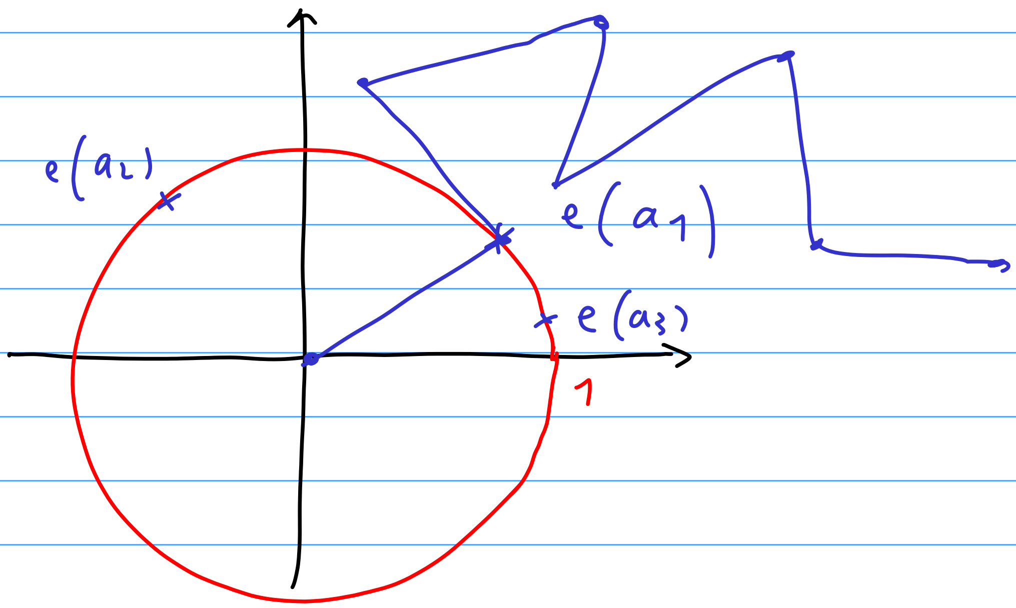
Exponential sums

1. Introduction

Exponential sums are expressions of the form

$$\sum_{n=1}^{N} e(a_n), \quad e(z) = e^{2i\pi z}$$

for some $\begin{cases} N \geq 1 \\ a_n \text{ in } \mathbb{R}. \end{cases}$



These are interesting because they show up a lot in number theory, especially analytic.

The goal for most applications is to get a non-trivial upper-bound, typically

$$\left| \sum_{n=1}^{N} e(an) \right| \leq N^{1-\delta}$$

for some $\delta > 0$, which means that the $e(an)$ "oscillate a lot".

This is a natural because: if the $e(an)$ are independent r.v., uniform on S^1, then

$$\sum_{n=1}^{N} e(an)$$

is for large N approximated by

a Gaussian with expectation 0 and variance

$$\sigma_N^2 = \sum_{n=1}^{N} 1 = N$$

In other words: <u>with very high probability</u>

$$\left| \sum_{n=1}^{N} e(a_n) \right| \approx \sqrt{N}$$

<u>Concrete example</u>: p prime

$$B(a;p) = \sum_{x=0}^{p-1} e\left(\frac{x^3 + ax}{p} \right) \qquad (\text{"Birch sums"})$$

<u>Theorem</u> (Weil)

For any p, a with $p \nmid a$, we have

$$|B(a;p)| \leq 2\sqrt{p}$$

(Instance of the Riemann Hyp. over finite fields)