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Theorem (Euler ; Dirichlet)

χ Dirichlet char. mod q

$\mathrm{Re}(s) > 1$

$$\sum_{n \geq 1} \frac{\chi(n)}{n^s} = L(s, \chi) = \prod_{p} \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

primes

Proof. For $M \geq 2$,

$$\prod_{p \leq M} \frac{1}{1 - \frac{\chi(p)}{p^s}} = \prod_{p \leq M} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}} + \dots\right)$$

modulus $\leq \frac{1}{p^{\mathrm{Re}(s)}} < 1$

$$= \prod_{p \leq M} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \dots\right)$$

$\chi(n)\chi(m) = \chi(nm)$

$$= \sum_{n \in D_M} \frac{\chi(n)}{n^s}$$

where $D_M = \{ n \geq 1 \mid$ all primes $p \mid n$ are $\leq M \}$

Formally : let $M \longrightarrow +\infty$,
(Appendix C) then D_M "$\longrightarrow$" $\{$all $n \geq 1\}$

so get

$$\prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$$

$\square$

<u>Remark</u>: analytic version of existence / uniqueness of prime factorization.

<u>Examples</u>:

(1) <u>$q = 1$</u> : one character mod 1 : $n \longmapsto 1$ for all n, so only one L-function :

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - 1/p^s}$$

(for $\mathrm{Re}(s) > 1$).

("Riemann zeta function")

(2) $q = 4$: two characters, so two L-functions

χ_0: $\chi_0(n) = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

so

$$L(s, \chi_0) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \dots$$

$$= \prod_p \frac{1}{1 - \frac{\chi_0(p)}{p^s}}$$

$$= \prod_{p \geq 3} \frac{1}{1 - \frac{1}{p^s}}$$

$$= \left(1 - \frac{1}{2^s}\right) \zeta(s)$$

In particular, the properties of

$L(s,\chi_0)$ follow from those of $\zeta(s)$.

[More generally for the "trivial character" mod $q \geqslant 1$, say ε_q, we get

$$L(s,\varepsilon_q) = \prod_{p|q}\left(1-\frac{1}{p^s}\right)\zeta(s)$$
]

.

$\underline{\chi_1}$: we can write

$$\begin{cases} \chi(2k+1) = (-1)^k \\ \chi(2k) = 0 \end{cases}$$

so

$$L(s,\chi_1) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots$$

$$= \prod_{p\equiv 1(4)} \frac{1}{1-\frac{1}{p^s}}$$

No very good properties!

$$\times \prod_{p\equiv 3(4)} \frac{1}{1+\frac{1}{p^s}}$$

Sketch of how to use Dirichlet L-functions to prove that

$$\boxed{\sum_{p \equiv a\,(q)} \frac{1}{p^\sigma} = \frac{1}{\varphi(q)} \sum_p \frac{1}{p^\sigma} + O(1)}$$

where $\sigma > 1$, $(a,q) = 1$, as $\sigma \longrightarrow 1$.

(Comparable to $\displaystyle\sum_{\substack{p \leq x \\ p \equiv a\,(q)}} \frac{1}{p} = \frac{1}{\varphi(q)} \sum_{p \leq x} \frac{1}{p} + O(1)$)

This also implies that

$$\pi(x; q, a) \xrightarrow[x \to +\infty]{} +\infty$$

for $(a,q) = 1$.

<u>Start with:</u>

$$\sum_{p \equiv a\,(q)} \frac{1}{p^\sigma} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \sum_p \frac{\chi(p)}{p^\sigma}$$

orthogonality relations

For any χ mod q, for $\sigma > 1$, we get from

$$\prod_p \frac{1}{1 - \frac{\chi(p)}{p^\sigma}} = \sum_{n \geq 1} \frac{\chi(n)}{n^\sigma}$$

the formula

$$-\sum_p \log\left(1 - \frac{\chi(p)}{p^\sigma}\right) = \log L(\sigma, \chi)$$

[here: need to be careful because we take logarithm of complex numbers; here we can define $\log(1-z)$ by the Taylor expansion around 0, for $|z| < 1$].

Using $-\log(1-z) = z + O(z^2)$ we get

$$\sum_p \frac{\chi(p)}{p^\sigma} + O(1) = \log L(\sigma, \chi)$$

as $\sigma \longrightarrow 1$.

[Ex. $q=1$, $\sum_p 1/p^\sigma + O(1) = \log \zeta(\sigma)$]

<u>Remark:</u> In <u>many</u> applications, it is the dependency with respect to q which is most important (e.g. Schoenberg's Th. for $p-1$)

Here we think q is <u>fixed</u>, so this is not an issue.

So we have:

$$\boxed{\sum_{p \equiv a\,(q)} \frac{1}{p^\sigma} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \log L(\sigma, \chi) + O(1)}$$

for $\sigma \longrightarrow 1$.

<u>Q.</u> How does $L(\sigma, \chi)$ behave as $\sigma \longrightarrow 1$?

We get

(1) bounded quantity if

$$L(\sigma, \chi) \longrightarrow \alpha \neq 0 \quad (\text{in } \mathbb{C})$$

(2) unbounded quantities

* <u>negative</u> : $L(\sigma, \chi) \longrightarrow 0$

* <u>positive</u> : $L(\sigma, \chi) \longrightarrow +\infty$

For <u>$\chi = \varepsilon_q$</u> :

$$L(\sigma, \varepsilon_q) = \prod_{p \mid q} \left(1 - \frac{1}{p^{\sigma}}\right) \zeta(\sigma)$$

$$\xrightarrow[\sigma \to 1]{} \prod_{p \mid q} \left(1 - \frac{1}{p}\right) = \frac{\varphi(q)}{q}$$

and $$\zeta(\sigma) = \sum_{n \geq 1} \frac{1}{n^{\sigma}} \sim \int_1^{+\infty} t^{-\sigma} dt = \frac{1}{\sigma - 1}$$

so $\log L(\sigma, \varepsilon_q) \xrightarrow[\sigma \to 1]{} +\infty$ (like $-\log(\sigma - 1)$)

so ε_q contributes to $\displaystyle\sum_{p \equiv a(q)} \frac{1}{p^\sigma}$

the quantity: $\log \zeta(\sigma) + O(1)$

$$\frac{1}{\varphi(q)} \underbrace{\varepsilon_q(a)}_{\substack{\shortparallel \\ 1}} \overbrace{\log\left(\frac{1}{\sigma-1}\right)}^{\substack{\log\zeta(\sigma) \\ \shortparallel}} + O(1)$$

1 because $(a,q) = 1$

as $\sigma \to 1$.

[Note: $\log\left(\frac{1}{\sigma-1}\right) = \log \zeta(\sigma) + O(1)$

$= \displaystyle\sum_p \frac{1}{p^\sigma} + O(1)$

by the previous computation for $q = 1$].

Ex. $q = 4$

$$\begin{cases} \chi_1(2k+1) = (-1)^k & \text{for } k \geq 0 \\ \chi_1(2k) = 0 \end{cases}$$

So $L(\sigma, \chi_1) = 1 - \frac{1}{3^\sigma} + \frac{1}{5^\sigma} - \dots$

exchange two limits, can be justified

$$\xrightarrow[\sigma \to 1]{} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$= \frac{\pi}{4} \quad (= \arctan(1))$$

$$\neq 0, \neq \infty$$

$\longrightarrow$ so $\log L(\sigma, \chi_1) = O(1)$ as $\sigma \to 1$

so

$$\sum_{p \equiv a\ (4)} \frac{1}{p^\sigma} = \frac{1}{2} \sum_p \frac{1}{p^\sigma} + O(1)$$

as $\sigma \to 1$

so there are infinitely many $p \equiv 1\ (4)$ and $p \equiv 3\ (4)$, in fact "about the same number of each".

Go back to general q:

Theorem (Dirichlet)

For $\chi \neq \varepsilon_q$, $L(\sigma, \chi) \xrightarrow[\sigma \to 1]{} L(1, \chi)$, which exists and is a non-zero complex number.

This implies:

$$\boxed{\sum_{p \equiv a\,(q)} \frac{1}{p^\sigma} = \frac{1}{\varphi(q)} \sum_{p} \frac{1}{p^\sigma} + O\left(1\right)}$$

as $\sigma \to 1$, for $(a,q)=1$.

To go beyond (towards $\pi(x;q,a)$) we use <u>complex analysis</u> and the <u>logarithmic-derivative</u> instead of the log.

Here observe that

$$L(s,\chi) \quad , \quad \mathrm{Re}(s) > 1$$

is holomorphic [uniformly convergent in compact subsets, terms $\frac{\chi(n)}{n^s}$ are holomorphic in $\mathbb{C}$], so we have an expansion of $\frac{L'}{L}$ in terms of primes.

Proposition – $q \geq 1$, $\chi \bmod q$

For $\mathrm{Re}(s) > 1$,

$$\boxed{-\frac{L'}{L}(s,\chi) = \sum_{n \geq 1} \Lambda(n)\chi(n)\, n^{-s}}$$

where the <u>von Mangoldt function</u> Λ is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^{\nu} \\ 0, & \text{unless } n = p^{\nu} \text{ for some prime } p \text{ and some } \nu \geq 1 \end{cases}$$

[Ex. $\Lambda(4) = \log(2)$]

<u>First approximation:</u> $\Lambda \approx \log \cdot$ (char. fion of primes)

Proof.
$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}, \quad \frac{(1/f)'}{(1/f)} = -\frac{f'}{f}$$

$\Rightarrow$ for $\mathrm{Re}(s) > 1$, we get

$$\frac{L'}{L}(s,\chi) = -\sum_{p} \log \operatorname{der} \operatorname{of}\left(1 - \frac{\chi(p)}{p^s}\right)$$

$$= -\sum_p \frac{-\chi(p)\cdot(-\log p)\,p^{-s}}{1-\frac{\chi(p)}{p^s}}$$

$$\boxed{\begin{aligned} p^{-s} &= \exp(-s\log p) \\ (p^{-s})' &= -(\log p)\,p^{-s} \end{aligned}}$$

$$= -\sum_p (\log p)\,\chi(p)\,p^{-s} \times \sum_{k\geq 0} \frac{\chi(p^k)}{p^{ks}}$$

$$= -\sum_p (\log p) \sum_{k\geq 1} \frac{\chi(p^k)}{p^{ks}}$$

$$= -\sum_n \frac{1}{n^s}\,\chi(n)\,\Lambda(n)$$

$\square$

So one is most naturally (from the <u>analytic</u> point of view) led to study

$$\boxed{\psi(x;q,a) = \sum_{\substack{n\leq x \\ n\equiv a\ (q)}} \Lambda(n)}$$

because

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi(a)}\, \psi(x;\chi)$$

where

$$\boxed{\psi(x;\chi) = \sum_{n \leq X} \Lambda(n)\chi(n).}$$

<u>Observe:</u>

$$\psi(x;q,a) = \sum_{\substack{n \leq x \\ n \equiv a\,(q)}} \Lambda(n)$$

$$= \sum_{\substack{p \leq x \\ p \equiv a\,(q)}} \log p + \sum_{k \geqslant 2} \sum_{\substack{p \leq x^{1/k} \\ p^k \equiv a\,(q)}} \log p$$

about primes; recall

$$\sum_{p \equiv a(q)} \frac{1}{p} = +\infty$$

quite small as $x \to +\infty$ because

$$\sum_{\substack{k \geqslant 2 \\ p^k \equiv a(q)}} \frac{1}{p^k} < +\infty$$

In fact the second term is

$$\le \underbrace{\frac{(\log x)}{(\log 2)}}_{\text{possibilities for } k \geq 2} \underbrace{(\log x)}_{\text{bound for } \log p} \underbrace{x^{1/2}}_{p^k \leq x \text{ has } \leq x^{1/k} \leq x^{1/2} \text{ solutions}}$$

whereas (as we will see)

$$\boxed{\psi(x; q, a) \underset{x \to \infty}{\sim} \frac{1}{\varphi(q)} x}$$

So

$$\sum_{\substack{p \leq x \\ p \equiv a \, (q)}} \log(p) \underset{x \to \infty}{\sim} \frac{1}{\varphi(q)} x$$

from which one deduces very easily that

$$\pi(x; q, a) \underset{x \to \infty}{\sim} \frac{1}{\varphi(q)} \frac{x}{\log(x)}$$

[because most primes will have $\log p \sim \log x$]

[C.5.6]

Theorem ("Explicit Formula")

Let $q \geqslant 1$, χ mod q Dirichlet character

$x \geqslant 2$, $X \geqslant 2$

$$\psi(x;\chi) = \sum_{n \le x} \Lambda(n)\,\chi(n)$$

$$= \delta_q\, x - \sum_{\substack{L(\beta+i\gamma,\chi)=0 \\ |\gamma| \le X,\ 0 \le \beta < 1}} \frac{x^{\beta+i\gamma}}{\beta+i\gamma} + O\left(\frac{(\log qx)^2 x}{X} + \log(x)\right).$$

$$\delta_q = \begin{cases} 1 & \text{if } \chi = \varepsilon_q \\ 0 & \text{if } \chi \neq \varepsilon_q \end{cases}$$

analytic continuation of $L(s,\chi)$