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Convention: p prime

Recall

Th. (Schoenberg, 1928) [1.4.1]

$N \geq 1$, $\Omega_N = \{1, \dots, N\}$, $\mathbb{P}_N$ uniform

$$F_N(n) = \frac{\varphi(n)}{n} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right), \quad \text{r.v. on } \Omega_N$$

Then $F_N \xrightarrow{\text{law}} F = \prod_p \left(1 - \frac{B_p}{p}\right)$

where (B_p) are independent Bernoulli r.v with

$$\mathbb{P}(B_p = 1) = \frac{1}{p}.$$

The product converges almost surely.

Remarks

(1) Non-generic limit!

In fact Erdős proved: the law of F is purely singular w.r.t Lebesgue measure: let

$$\Phi(t) = \mathbb{P}(F \leq t)$$

then Φ is strictly increasing, continuous, and differentiable almost everywhere with $\Phi'(t) = 0$ a.e.

(2) This shows that $\varphi(n)$ is complicated; another illustration is the following: if ℓ is any fixed prime number, then "almost all $\varphi(n)$ are divisible by ℓ", in the sense that

$$\lim_{N \to \infty} \frac{1}{N} \left| \{ 1 \leq n \leq N \mid \ell \mid \varphi(n) \} \right| = 1.$$

Motivation for the result

$$\frac{\varphi(n)}{n} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right) = \prod_{p} \left(1 - \frac{\underline{B}_p(n)}{p}\right)$$

where $\underline{B}_p : \Omega_N \longrightarrow \{0, 1\}$

s.t. $\underline{B}_p(n) = 1 \iff p \mid n$

underline : arithmetic r.v

Observe: $\left(\underline{B}_p\right)_p \xrightarrow[N\to\infty]{\text{law}} \left(B_p\right)$

(in the sense that for $M \geq 1$,

$$\left(\underline{B}_p\right)_{p\leq M} \xrightarrow{\text{law}} \left(B_p\right)_{p\leq M}\Big)$$

as a consequence of the Cor of Th. 1.3.1:

$$\left\{\underline{B}_p(n) = \varepsilon_p,\ p\leq M\right\}$$

$$= \left\{ n \bmod p = \begin{cases} 0, & \varepsilon_p = 1 \\ \text{not}, & \varepsilon_p = 0 \end{cases},\ p\leq M \right\}$$

$$\xrightarrow{\text{Cor.}} \prod_{p\leq M} \frac{1}{p}\left(\text{nb. of } x \bmod p \text{ congruent to } \begin{cases} 0 \\ \text{not} \end{cases}\right)$$

$$= \prod_{p\leq M} \mathbb{P}\left(B_p = \varepsilon_p\right)$$

Now

$$F_N = \prod_p \left(1 - \frac{\underline{B}_p}{p}\right)$$

We expect that

$$F_N \xrightarrow{law} \prod_p \left(1 - \frac{B_p}{p}\right)$$
$$\shortparallel$$
$$F$$

This is not yet a proof because the product is infinite; indeed one has

$$\sum_p \frac{1}{p} = \sum_p \mathbb{P}(B_p = 1)$$
$$= +\infty \quad (\text{Euler})$$

Borel-Cantelli
$\Rightarrow$ a.s. $B_p = 1$ for infinitely many primes

We need to justify exchanging two limits.

Probabilistic tool

Prop. [B.4.4]

Let $M = \mathbb{R}$ (or $\mathbb{R}^d$)

Let $(X_n)_{n \geq 1}$ M-valued r.v

Let $X_{n,m}$ ______

$m \geq 1$, $n \geq 1$

and $X_n = X_{n,m} + E_{n,m}$.

Assume:

(1) $X_{n,m} \xrightarrow[n \to \infty]{\text{law}} Y_m$ for fixed m

(2) $\lim_{m \to \infty} \overline{\lim_{n \to \infty}} \mathbb{E}(|E_{n,m}|) = 0$.

Then (X_n), and (Y_m), converge in law and have the same limit.

Proof of Schoenberg's Th.

Apply Prop. B.4.4 with

$$X_n \longleftrightarrow \underline{F}_N = \prod_p \left(1 - \frac{B_p}{p}\right)$$

$$X_{n,m} \longleftrightarrow \underline{F}_{N,M} = \prod_{p \le M} \left(1 - \frac{B_p}{p}\right)$$

(1) $$\underline{F}_{N,M} \xrightarrow[N \to \infty]{\text{law}} \prod_{p \le M} \left(1 - \frac{B_p}{p}\right) = Y_M$$

"mapping principle"

(2)

$$\mathbb{E}_N\left(\left|\underline{E}_{N,M}\right|\right) = \frac{1}{N} \sum_{1 \le n \le N} \left| \prod_{p \mid n} \left(1 - \frac{1}{p}\right) - \prod_{\substack{p \mid n \\ p \le M}} \left(1 - \frac{1}{p}\right) \right|$$

$$\le \frac{1}{N} \sum_{1 \le n \le N} \underbrace{\left| \prod_{\substack{p \mid n \\ p > M}} \left(1 - \frac{1}{p}\right) - 1 \right|}$$

say $\{p > M,\ p \mid n\} = \{p_1, \dots, p_r\}$

then the difference is

$$\left| \cancel{1} - \frac{1}{p_1} - \dots - \frac{1}{p_r} + \frac{1}{p_1 p_2} + \dots - \cancel{1} \right|$$

$$\leq \sum_{\substack{M < d \leq N \\ d \mid n}} \frac{1}{d}$$

so

$$\mathbb{E}_N(|E_{N,M}|) \leq \frac{1}{N} \sum_{n \leq N} \sum_{\substack{M < d \leq N \\ d \mid n}} \frac{1}{d}$$

$$\leq \sum_{M < d \leq N} \frac{1}{d} \frac{1}{N} \sum_{\substack{n \leq N \\ n \equiv 0 (d)}} 1$$

$$\leq \sum_{M < d} \frac{1}{d^2}$$

$$\Longrightarrow \quad \overline{\lim} \ \mathbb{E}_N(|E_{N,M}|) \leq \sum_{M < d} \frac{1}{d^2}$$

so $\lim_{M\to\infty} \overline{\lim} \left(\_\_\_ \right) = 0$,

as we wanted!

Prop. B.4.4 applies

$\Rightarrow$ $\underline{F}_N$ and $Y_M = \prod_{p \in M} \left(1 - \frac{B_p}{p}\right)$

converge in law with same limit,

which is $\prod_p \left(1 - \frac{B_p}{p} \right)$.

Last step: product converges a.s.

This follows from Kolmogorov's "3-series Theorem"

<u>Th.</u> (B.10.1)

(X_n) $\mathbb{R}$-valued independent

If both $\begin{cases} \sum \mathbb{V}(X_n) \\ \sum \mathbb{E}(X_n) \end{cases}$ converge

$\left[\mathbb{V}(X) = \mathbb{E}\left(|X - \mathbb{E}(X)|^2\right) \right.$

$X \in L^2(\Omega)$ $\left. = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \right]$

Then $\sum X_n$ converges almost surely.

Here

$$\log F = \sum \underbrace{\log\left(1 - \frac{B_p}{p}\right)}_{X_p}$$

where (X_p) are independent

$$\mathbb{E}(X_p) = \frac{1}{p} \log\left(1 - \frac{1}{p}\right) \sim -\frac{1}{p^2}$$

$$\mathbb{V}(X_p) = \frac{1}{p} \log\left(1 - \frac{1}{p}\right)^2 - \mathbb{E}(X_p)^2$$

$$\sim \frac{1}{p^3}$$

so the assumptions hold.

$\square$ (proof of Schoenberg's Th.)

Now we explain the proof of B. 4. 4,

We use :

(1) In the def. of convergence in law :

$\forall f \in \underbrace{C_b(M)}_{\text{cont. bounded}}$, $\mathbb{E}(f(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(f(X))$

we can restrict f to be

1a) Lipschitz on M, i.e.

$$|f(x)-f(y)| \leq C\, d(x,y)$$

for some $C \geq 0$ [$\Rightarrow$ unif^{ly} continuous]

1b) Smooth compactly supported f of $M = \mathbb{R}$ (or $\mathbb{R}^d$)

(2) Th. (B.1.1, "Riesz-Markov Theorem")

M locally compact [$\mathbb{R}$, $\mathbb{R}^d$]

A (Radon) probability measure μ on M is "the same" as a linear map $\Lambda : \underbrace{C_c(M)}_{\text{cont., compact support}} \longrightarrow \mathbb{C}$

such that $\Lambda(f) \geq 0$ for $f \geq 0$

and $\sup_{\substack{0 \leq f \leq 1 \\ f \in C_c(M)}} \Lambda(f) = 1.$

"The same" means that any such Λ is of form

$$\Lambda(f) = \int_M f(x)\, d\mu(x)$$

for a unique Radon prob. measure.

If $M = \mathbb{R}$ [or $\mathbb{R}^d$] then one can restrict Λ to $C_c^\infty(M)$.

Proof of B.4.4. $M = \mathbb{R}$

X_n, $X_{n,m}$, $E_{n,m}$ as there

Let $f \in C_c^\infty(\mathbb{R})$

Step 1. $\mathbb{E}(f(X_n))$ converges to some limit $\Lambda(f)$

We may assume $f : \mathbb{R} \longrightarrow \mathbb{R}$.

The function f is Lipschitz:

$$|f(x) - f(y)| \le C\,|x-y|$$

$\Rightarrow$

$$|f(X_n) - f(X_{n,m})| \le C\,|E_{n,m}|$$

$\Rightarrow$

$$\left|\mathbb{E}(f(X_n)) - \mathbb{E}(f(X_{n,m}))\right| \le C\,\mathbb{E}(|E_{n,m}|)$$

or

$$-C\,\mathbb{E}(|E_{n,m}|) + \mathbb{E}(f(X_{n,m}))$$

$$\le \mathbb{E}(f(X_n)) \le \mathbb{E}(f(\underbrace{X_{n,m}}_{\xrightarrow{\text{law}} Y_m}))$$

$$+ C\,\mathbb{E}(|E_{n,m}|)$$

Fix $m \geq 1$, let $n \longrightarrow \infty$:

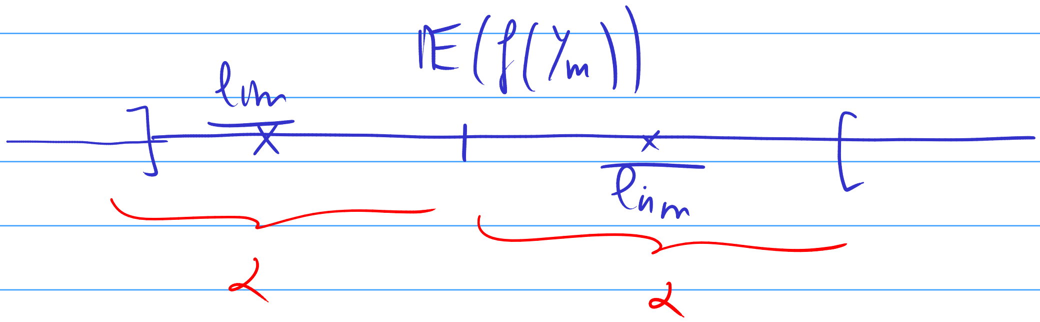
$$-C\,\overline{\lim}\,\mathbb{E}(|E_{n,m}|)$$

$$+ \mathbb{E}(f(Y_m)) \le \underline{\lim}\,\mathbb{E}(f(X_n))$$

$$\le \overline{\lim}\,\mathbb{E}(f(X_n))$$

$$\le \mathbb{E}(f(Y_m))$$

$$\underbrace{+\, C\,\overline{\lim_{n\to\infty}}\,\mathbb{E}(|E_{n,m}|)}_{\alpha}$$



$$\Rightarrow \quad 0 \leq \overline{\lim} - \underline{\lim} \leq 2C \ \overline{\lim}_{n\to\infty} \mathbb{E}(|E_{n,m}|)$$

for every $m \geq 1$.

$$\text{Assumption (2)} \iff \left(\text{R.H.S} \xrightarrow[m\to\infty]{} 0 \right)$$

so $\quad \overline{\lim} = \underline{\lim}$

so $\quad \mathbb{E}(f(X_n))$ <u>converges</u>.

<u>Step 2.</u> Λ "is" a Radon prob. measure

Indeed, Λ is linear is clear,

and also $\begin{cases} \Lambda(f) \geq 0 & \text{if } f \geq 0 \\ \Lambda(1) = 1 \end{cases}$

so B.1.1 implies that

$$\Lambda(f) = \int f \, d\mu$$

for some μ prob. on $\mathbb{R}$, and then Step 1 shows that

$$X_n \xrightarrow{\text{law}} \mu .$$

Step 3. Also $Y_m \left[= \lim_{n\to\infty} X_{n,m} \right]$ converges to μ: $f \in C_c^\infty(\mathbb{R})$, inequalities above give

$$-C \overline{\lim} \, \mathbb{E}(|E_{n,m}|) + \mathbb{E}(f(Y_m)) \le \underset{=\,\Lambda(f)}{\lim \mathbb{E}(f(X_n))} \le \mathbb{E}(f(Y_m)) + C \overline{\lim_{n\to\infty}} \, \mathbb{E}(|E_{n,m}|)$$

$$\Rightarrow \quad \left| \mathbb{E}(f(Y_m)) - \Lambda(f) \right| \le C \overline{\lim_{n\to\infty}} \, \mathbb{E}(|E_{n,m}|)$$

for every $m \geq 1$.

Let $m \to \infty$, apply (2), so RHS tends to 0, so $\mathbb{E}(f(Y_m)) \xrightarrow[m\to\infty]{} \Lambda(f)$.

This gives exactly

$$Y_m \xrightarrow{\text{law}} \mu$$

$\square$ [B.4.4]