25.3.2021

Explicit formula

$$\psi(x;\chi) = \delta_q x - \sum_{\substack{\rho \text{ zero} \\ \text{of } L(s,\chi) \\ 0 \le \mathrm{Re}\,\rho \le 1 \\ |\mathrm{Im}\,\rho| \le x}} \frac{x^\rho}{\rho} + (\text{error term})$$

$$\psi(x;\chi) = \sum_{n \le x} \Lambda(n)\chi(n)$$

counted with multiplicity

Idea for the existence of such a relation (Riemann)

$$\sum_{n \le x} \Lambda(n)\,\chi(n) = \sum_{n \ge 1} \Lambda(n)\,\chi(n)\,\varphi\left(\frac{n}{x}\right)$$

where φ = characteristic function of $[0,1]$

Variant of Plancherel formula:

$$\sum_{n \in \mathbb{Z}} a_n \overline{b_n} = \int_0^1 f(t)\,\overline{g(t)}\,dt$$

a_n = Fourier coeff. of f

b_n = Fourier coeff. of g

However instead of periodic functions and Fourier series we need to use "multiplicative" versions: Dirichlet series / Mellin transforms.

<u>Definition</u>:

(1) $(a_n)_{n \geq 1}$ complex numbers

$$D(s) = \sum_{n \geq 1} a_n n^{-s}, \quad s \in \mathbb{C}$$

is the associated <u>Dirichlet series</u> (wherever it makes sense).

(2) For $f : [0, +\infty[\longrightarrow \mathbb{C}$

The Mellin transform $\hat{f}(s)$ is defined by

$$\hat{f}(s) = \int_0^{+\infty} f(t)\, t^s \frac{dt}{t}$$

(wherever it makes sense).

Ex: (1) $a_n = 1 \longrightarrow D(s) = \zeta(s)$

$a_n = \chi(n) \longrightarrow D(s) = L(s,\chi)$

$a_n = \chi(n)\Lambda(n) \longrightarrow D(s) = -\frac{L'}{L}(s,\chi)$

(2) $f = \mathbb{1}_{[0,1]}$

$$\longrightarrow \hat{f}(s) = \int_0^1 t^s \frac{dt}{t} = \frac{1}{s}$$

(for $Re(s) > 0$).

$f(t) = e^{-t}$

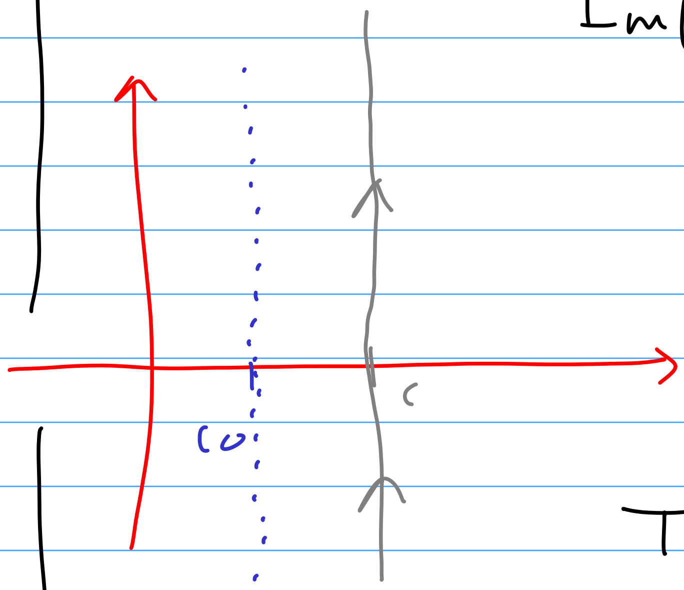
$$\longrightarrow \hat{f}(s) = \int_0^{+\infty} e^{-t} t^s \frac{dt}{t}$$

$$= \Gamma(s) \quad \text{(Gamma function).}$$

[A.3.1]

<u>Prop. (Mellin inversion)</u> - Suppose f is such the $\hat{f}(s)$ exists for $Re(s) > c_0$ and decays fast enough when $Re(s) \geq c_0 + \varepsilon$ and

$\mathrm{Im}(s) \longrightarrow +\infty$



(**Note:** $|t^s| = t^{\mathrm{Re}(s)}$)

Then for <u>any</u> <u>fixed</u> $c > c_0$, and $t > 0$ we have

$$f(t) = \frac{1}{2i\pi} \int_{(c)} \hat{f}(s)\, t^{-s}\, ds$$

vertical line $\mathrm{Re}(s) = c$, oriented $\uparrow$

modulus t^{-c}

(<u>Idea:</u> write

$$\hat{f}(s) = \int_0^{+\infty} f(t)\, t^s \, \frac{dt}{t}$$

Put $t = e^y$: $dt = e^y dy$

so $\dfrac{dt}{t} = dy$

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(e^y)\, e^{ys}\, dy$$

Write $s = \sigma + it$, $\sigma = \mathrm{Re}(s)$:

$$\hat{f}(\sigma + it) = \int_{-\infty}^{+\infty} f(e^y)\, e^{\sigma y}\, e^{ity}\, dy.$$

(σ fixed)
t variable

= Fourier transform of $g(y) = f(e^y)\, e^{\sigma y}$

So provided Fourier inversion applies we get :

$$g(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\sigma + it)\, e^{-ity}\, dt$$

Put $e^y = x$:

$$f(x)\, x^{\sigma} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\sigma + it)\, x^{-it}\, dt$$

$$\iff \boxed{f(x) = \frac{1}{2i\pi} \int_{\mathbb{R}} \hat{f}(s)\, x^{-s}\, ds} \quad [s = \sigma + it]$$

Note: For $f = \mathbb{1}_{[0,1]}$, the decay of $\hat{f}(s)$ is not sufficient to really apply this formula!

Consider:

$$\sum_{n\geq 1} a_n f\left(\frac{n}{x}\right)$$

for $\begin{cases} (a_n) \text{ arbitrary, bounded} \\ f \text{ a nice function} \end{cases}$

(eg. $f \in C_c^\infty(]0,+\infty[)$)

Then by Mellin inversion we get

$$f\left(\frac{n}{x}\right) = \frac{1}{2i\pi}\int_{(c)} \hat{f}(s)\left(\frac{n}{x}\right)^{-s} ds$$

so (assuming exchange of sum and integral is allowed)

$$\sum_{n\geq 1} a_n f\left(\frac{n}{x}\right) = \frac{1}{2i\pi}\int_{(c)} \hat{f}(s)\, x^s \left(\sum_{n\geq 1} a_n n^{-s}\right) ds$$

i.e.

$$\boxed{\sum_{n\geq 1} a_n f\left(\frac{n}{x}\right) = \frac{1}{2i\pi}\int_{(c)} \hat{f}(s)\, D(s)\, x^s \, ds}$$

In our case: $a_n = \Lambda(n)\, \chi(n)$

$$\sum_{n\leq x} \chi(n)\, \Lambda(n) \text{ "="} \frac{1}{2i\pi}\int_{(c)} \frac{1}{s}\left(-\frac{L'}{L}(s,\chi)\right) x^s \, ds$$

the integral does not converge absolutely

Q. How to see on the <u>integral side</u> the size / asymptotic of the LHS?

If we try a direct estimate we have

$$\left| \hat{f}(s)\, D(s)\, x^s \right| = \left|\hat{f}(s)\, D(s)\right| x^c$$

for $\mathrm{Re}(s) = c$ so "triangle inequality" is

$$\left| \sum_{n \le x} a_n f\left(\frac{n}{x}\right) \right| \le \frac{x^c}{2\pi} \underbrace{\int_{-\infty}^{+\infty} |\hat{f}(c+it)\, D(c+it)|\, dt}_{\text{just a number (if convergent)}}$$

But if a_n is of size $\simeq 1$ the series $D(s)$ only makes sense for $\mathrm{Re}(s) > 1$, so one needs to take $c > 1$, but the triangle inequality is (typically)

$$\left| \sum_{n \le x} a_n f\left(\frac{n}{x}\right) \right| \le C x < \frac{x^c}{2\pi}$$

(x large)

Look now at $a_n = \chi(n) \Lambda(n)$

so $D(s) = -\frac{L'}{L}(s)$ and we

are looking at

$$\frac{1}{2i\pi}\int_{(c)} \hat{f}(s)\left(-\frac{L'}{L}(s,\chi)\right)x^s\,ds$$

$(c>1)$

But $\hat{f}(s)$ is often meromorphic

[e.g. $\hat{f}(s)=\frac{1}{s}$,

$\hat{f}(s)=\Gamma(s)$]

and $-\frac{L'}{L}(s,\chi)$ is holomorphic where defined by the series, and one can prove:

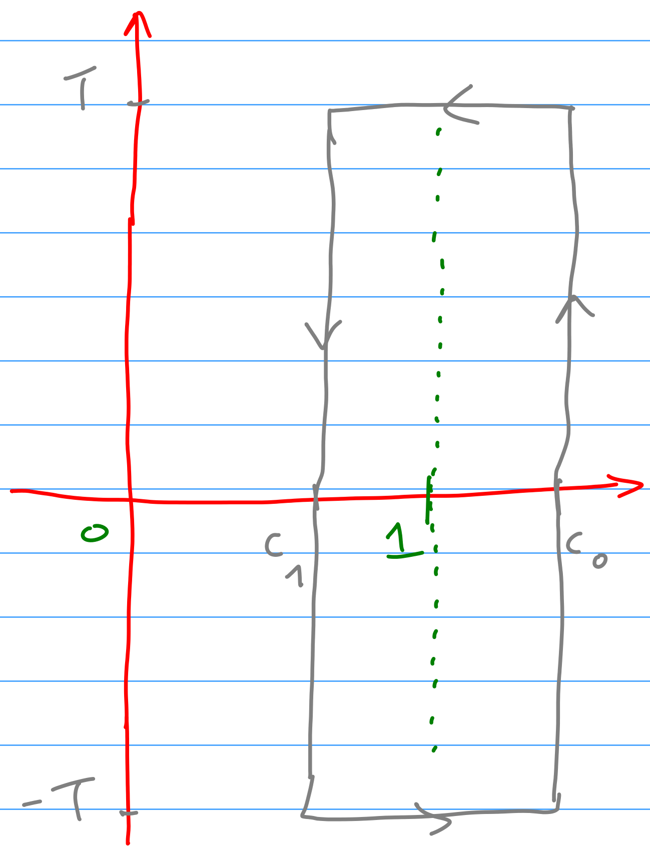
Th. The L-functions $L(s,\chi)$ are meromorphic (by analytic continuation) on $\mathbb{C}$, in fact with no poles at all unless $\chi=\varepsilon_q$, in which case there is a single simple pole at $s=1$ (with residue $\varphi(q)/q$).

(Simple proofs for continuation to $\mathrm{Re}(s) > 0$, see later for $\zeta(s)$).

So $-\frac{L'}{L}(s, \chi)$ is meromorphic on $\mathbb{C}$ with:

(1) a simple pole with residue 1 at $s = 1$ if $\chi = \varepsilon_q$

(2) simple poles at any zero ρ of $L(s, \chi)$ with residue equal to minus $\underbrace{\text{the multiplicity of the zero}}_{m_\chi(\rho)}$.



Applying Cauchy's Theorem we get

$$\frac{1}{2i\pi} \int \hat{f}(s) - \frac{L'}{L}(s,\chi)\, x^s \, ds$$

no pole

entire

$$= \overbrace{\delta_q \, \hat{f}(1)\, x}^{\text{pole at } s=1 \text{ for } \chi = \varepsilon_q} - \sum_{\substack{\text{zeros of} \\ L(s,\chi) \text{ in } \square}} m_\chi(\rho)\, \hat{f}(\rho)\, x^\rho$$

$c_1 > 0$, $\hat{f}$ has no pole for $\mathrm{Re}(s) > 0$

[e.g. $\frac{1}{s}$, $\Gamma(s)$]

For $f = \mathbb{1}_{[0,1]}$:

$$\frac{1}{2i\pi} \int_{\square} \frac{1}{s} \cdot \left(- \frac{L'}{L}(s,\chi) \right) x^s \, ds$$

$$= \boxed{\delta_q\, x - \sum_{\text{zeros}} m_\chi(\rho) \frac{x^\rho}{\rho}}$$

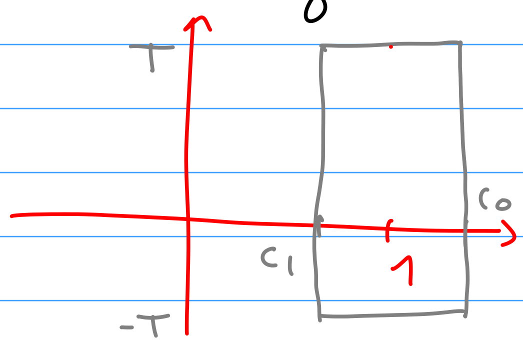
main part of RHS of the explicit formula!

To get the explicit formula, one needs :

(1) no zero has real part > 1

(because $L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$ for $\mathrm{Re}(s) > 1$, and the product converges absolutely with no term equal to 0)

(2) handle the horizontal contributions



and show they go to zero as $T \to +\infty$

(3) estimate the integral over $\mathrm{Re}(s) = c_1$:

$$\left| \frac{1}{2i\pi} \int_{c_1 - iT}^{c_1 + iT} \hat{f}(s) \left(-\frac{L'}{L}(s,\chi) \right) x^s \, ds \right|$$

$$\leq \frac{x^{c_1}}{2\pi} \underbrace{\int_{-\infty}^{+\infty} \left| \hat{f}(c_1+it) - \frac{L'}{L}(c_1+it) \right| dt}_{\text{(assume convergent)}}$$

(4) Let $c_1 \longrightarrow 0$.

This "explains" the explicit formula.

<u>Corollary</u>: <u>If</u> all zeros ρ in $0 \leq \mathrm{Re}(s) \leq 1$ have $\mathrm{Re}(\rho) = 1/2$

then

$$\psi(x;\chi) = \delta_q x + O\left(x^{1/2} (\log qx)^2\right)$$

<u>Def.</u> <u>GRH(q)</u> is the statement that the zeros of any $L(s,\chi)$ with χ mod q with real part

between 0 and 1 have real part $\frac{1}{2}$.

If GRH(q) holds: by orthogonality

$$\boxed{\begin{aligned}\psi(x;q,a) &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)}\, \psi(x;\chi) \\ &= \frac{x}{\varphi(q)} + O\left(x^{1/2} (\log qx)^2 \right)\end{aligned}}$$

explains the normalization in

$$\underline{N}_x(x) = \frac{\log x}{\sqrt{x}} \left(\varphi(q)\, \pi(x;q,a) - \pi(x) \right)$$

Cor. (de la Vallée Poussin, Hadamard)
1896 – 190? $\qquad q=1$

$$\psi(x;\chi) = \delta_q\, x + O\left(\frac{x}{(\log x)^A} \right)$$

for any $A > 0$ (for q fixed).

In particular, it is easy to deduce

Th. [Prime Number Theorem] 1896

$$\pi(x) = \int_2^x \frac{dt}{\log(t)} + O\left(\frac{x}{(\log x)^A}\right)$$

$$li(x) \underset{x\to\infty}{\sim} \frac{x}{\log x}$$