26.4.2021

Q. How to get numerical information out of this formula for $N_4(3) - N_4(1)$?

The expression as a series with independent terms gives a formula for the <u>characteristic function</u> of $N_4(3) - N_4(1) = D_4$:

$$\mathbb{E}\left(e^{itD_4}\right) = e^{2it} \prod_{\gamma>0} e^{4it\,\mathrm{Re}\left(\frac{I_\gamma}{1/2+i\gamma}\right)}$$

$(t \in \mathbb{R})$

by independence

$$= e^{2it} \prod_{\gamma>0} J_0\left(\frac{4t}{\sqrt{1/4+\gamma^2}}\right)$$

where J_0 is a <u>Bessel function</u>, namely

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos(t)}\,dt$$

($\simeq$ Fourier transform of I_j)

These functions are very-well understood, and one deduces for instance that

$$f(t) = \mathbb{E}\left(e^{itD_4}\right)$$

is smooth, and then by Fourier analysis that D_4 has a (smooth) density with respect to Lebesgue measure: law of D_4 is

$$g_4(x)\,dx$$

for some function g_4 (in fact

$$g_4(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)\,e^{-itx}\,dt$$

by Fourier inversion).

Then for instance

$$\mathbb{P}(D_4 > 0) = \int_0^{+\infty} g_4(x)\,dx.$$

$$\mathbb{P}(N_4(3) - N_4(1))$$

Knowing numerically "many" zeros ρ of $L(s, \chi_4)$, one first sees that they do satisfy $\mathrm{Re}(\rho) = \frac{1}{2}$, one can compute $f(t)$ approximately, then by integration one can compute this $\underbrace{\text{approximately}}_{\text{probability}}$ (with error goods):

$$\mathbb{P}(D_4 > 0) = 0.99\ldots$$

One can also use the series expression for N_4 to compute estimates for the "tails" of D_4:

$$? \leq \mathbb{P}(\pm D_4 > T) \leq ?.$$

[5.4.8]

Proposition

There exist constants $c_1, c_2 > 0$ such that

$$c_2 \exp\left(-\exp\left(c_2\sqrt{T}\right)\right) \le \mathbb{P}\left(\pm(D_4 - 2) > T\right) \le c_1 \exp\left(-\exp\left(\frac{\sqrt{T}}{c_1}\right)\right)$$

for T large.

Idea:

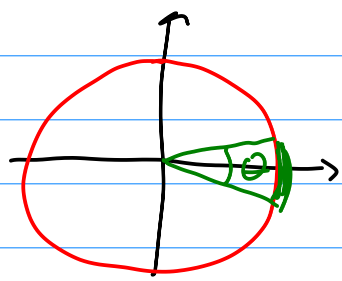
$$D_4 - 2 = 4 \sum_{\gamma > 0} \mathrm{Re}\left(\frac{I_\gamma}{1/2 + i\gamma}\right)$$

$$= 4\,\mathrm{Re}\left(\sum_{\gamma>0} \frac{I_\gamma}{1/2 + i\gamma}\right)$$

does not converge absolutely

so sometimes we get large values because we can ensure with small, >0, probability that

$$\boxed{I_\gamma \simeq 1}$$



for the first γ's (say $\gamma \leq M$)

$\longrightarrow$

$$\mathrm{Re}\left(\sum_{0<\gamma\leq M} \frac{I_\gamma}{1/2+i\gamma} \right) \simeq \mathrm{Re}\left(\sum_{|\gamma|\leq M} \frac{1}{1/2+i\gamma} \right)$$

with probability

$$\simeq \left(\frac{2\theta}{2\pi} \right)^{(\text{nb. of } \gamma < M)}$$

Then the remainder of the series is actually symmetric, so takes a values ≥ 0 with probability $\geq 1/2$

$$\longrightarrow \quad \mathbb{P}\left(D_4 - 2 > T \right) \geq \frac{1}{2} \left(\frac{\theta}{\pi} \right)^{(\text{nb. of } \gamma < M)}$$

for M (roughly) chosen so that

$$\mathrm{Re}\left(\sum_{\gamma < M} \frac{1}{1/2+i\gamma} \right) > T.$$

One can optimize then the value of M (and θ) ....

For the <u>upper bound</u> for

$$\mathbb{P}(D_4 - 2 > T)$$

one uses

(i) (sub) gaussian behavior:

$$\mathbb{P}(D_4 - 2 > T) \leq \exp(-\alpha T^2)$$

$$(\text{for some } \alpha > 0)$$

[comes from having a series of independent terms]

(ii) combined with the fact that the individual summands $\frac{I_y}{1/2 + iy}$ are <u>bounded</u>.

← the series is far from a Gaussian series.

[For <u>details</u>: see chapter on exponential sums]

[Notes: chapter 3 (+ 4)]

Chapter IV

The values of $\zeta(s)$

1. Introduction

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad Re(s) > 1$$

$$= \prod_p \frac{1}{1 - 1/p^s}, \quad Re(s) > 1$$

We saw that ζ has meromorphic to $\mathbb{C}$, with a simple pole at $s=1$ with residue 1.

We saw that $\zeta(s)$ is very important in understanding the distribution of primes: by the explicit formula

$$\psi(x) = \sum_{n\leq x} \Lambda(n)$$

von Mangold function

$$= x - \sum_{\substack{\zeta(\beta+i\gamma)=0 \\ 0<\beta<1 \\ |\gamma|\leq T}} \frac{x^\rho}{\rho} + (\text{error})$$

(so the zeros of ζ "control" the number of primes $\leq x$).

In particular: if all zeros ρ above have $\beta = \mathrm{Re}(\rho) = \frac{1}{2}$ [RH]

then

$$\psi(x) = x + O\left(\sqrt{x}\,(\log x)^2\right)$$

for $x \geq 2$.

[Application: if $\frac{1}{2} < \theta < 1$, RH implies that there primes p with

$$x < p < x + x^\theta$$

if x large enough, because

$$(*) \quad \psi(x + x^\theta) - \psi(x) = x^\theta + O\left(\sqrt{x}\,(\log x)^2\right)$$

(and there are not enough prime powers to account for x^{θ})

<u>One consequence</u> (due to Mill):

There exists $\alpha \geq 1$ s.t.

$\lfloor \alpha^{3^n} \rfloor$ is a prime

for <u>all</u> $n \geq 1$.

(<u>unconditional</u> because Hoheisel proved existence of $\theta < 1$ with (*))

<u>Q.</u> What are the <u>statistical properties</u> of $\zeta(s)$ as s varies (in some region)?

This depends on the region:

(1) <u>$\mathrm{Re}(s) > 1$</u> : "<u>easy</u>" because the series converges absolutely.

(2) <u>$1/2 < \mathrm{Re}(s) \leq 1$</u> : no absolute convergence, but there should be no zeros,

and we will obtain a functional limit theorem (Bagchi's Theorem)

(in an ∞-dimensional space)

(3) $\operatorname{Re}(s) = \frac{1}{2}$: "the critical line"

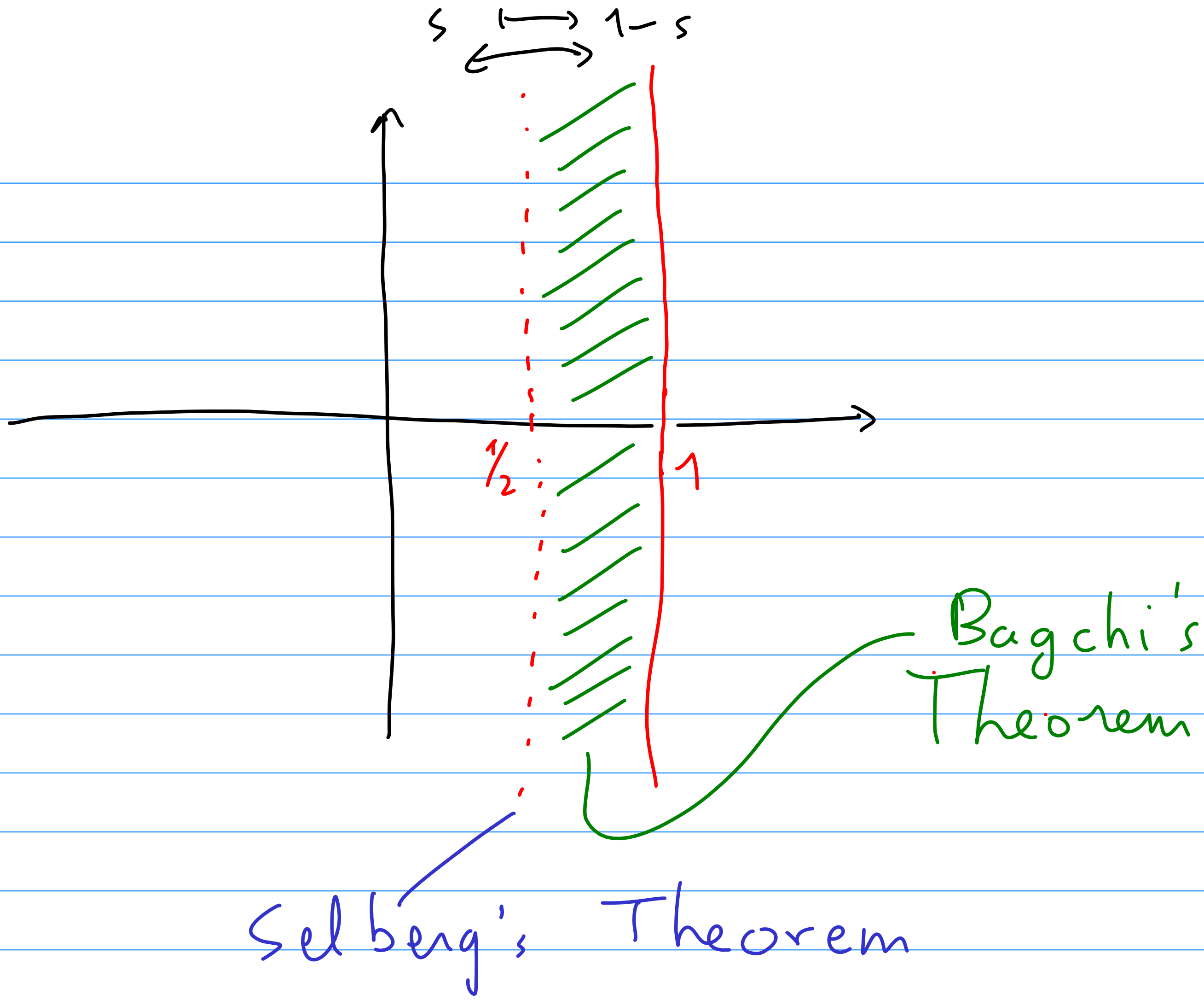
$\rightsquigarrow$ very different behavior than Bagchi's Theorem ("Selberg's Central Limit Theorem")

(4) $\operatorname{Re}(s) < \frac{1}{2}$: predictable because of the "functional equation":

$$\pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\,\Gamma\!\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$(s \in \mathbb{C} - \{0,1\})$

(guessed by Euler; proved by Riemann).



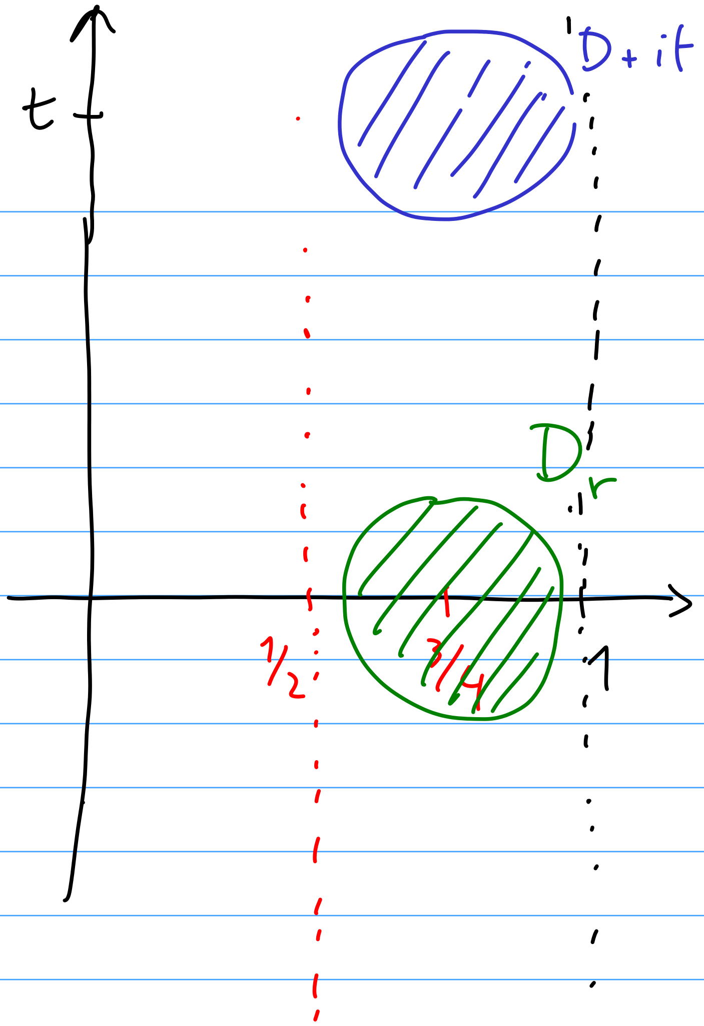
Bagchi's Theorem

Let $0 < r < \frac{1}{4}$, and

$D = D_r =$ open disc in $\mathbb{C}$ with center at $\frac{3}{4} \in \mathbb{R}$, with radius r

(so $D_r \subset \{ s \mid \frac{1}{2} < \mathrm{Re}(s) < 1 \}$)

$\mathcal{H}(D) = \{ f : \overline{D} \to \mathbb{C} \ /$
f continuous on $\overline{D}$,
f holomorphic on $D \}$



$\mathcal{H}(D)$ is an infinite-dimensional separable complex Banach space with norm

$$\|f\| = \sup_{z \in \overline{D}} |f(z)| = \sup_{z \in \partial D} |f(z)|$$

maximum principle

$\partial D =$ circle centered at $3/4$ with radius $\wedge$

Theorem (Bagchi, 1983)

For $T > 0$, define

$$\Omega_T = [-T, T]$$

$$\mathbb{P}_T = \frac{1}{2T}\, dt$$

Define random variables

$$\underline{Z}_T : \Omega_T \longrightarrow \mathcal{H}(D)$$

by mapping $t \in \Omega_T$ to the holomorphic function

$$s \longmapsto \underbrace{\zeta(s+it)}_{\substack{\text{hol. on} \\ \{s \mid s+it \neq 1\} \\ \shortparallel \\ \{s \mid s \neq 1-it\}}}$$

for $s \in \overline{D}$.

("Vertical translates of ζ on $\overline{D}$")

in particular on D

Then $\underline{Z}_T$ converge in law as $T \to +\infty$ to some (explicit) random holomorphic function on D.

"Corollary" (Voronin's Th., 1975 "Universality Theorem")

Let $f \in \mathcal{H}(D)^\times$ [i.e. f has no zeros on $\overline{D}$, e.g. $f(s) = e^s$]

Let $\varepsilon > 0$.

Then

$$\liminf_{T\to\infty} \frac{1}{2T} \underset{\text{Lebesgue}}{\lambda} \left(\left\{ t \in \Omega_T \,\middle|\, \sup_{s\in\overline{D}} |f(s) - \zeta(s+it)| < \varepsilon \right\} \right)$$

$$> 0.$$

How does one derive the corollary?

[B.3.3]

<u>Lemma.</u> Let M be a <u>separable</u> complete metric space (e.g. $\mathcal{H}(D)$).

Let X_n be M-valued r.v. converging in law to X.

Let S be the <u>support</u> of X [i.e. the complement in M

of the union of all open sets U s.t. $\mathbb{P}(X \in U) = 0$], a closed subset of M.

Then for any $x \in S$ and any open neighb. V of x, we have

$$\liminf_{n \to \infty} \mathbb{P}(X_n \in V) > 0.$$

Take $\begin{cases} M = \mathcal{H}(\mathbb{D}) \\ X_n = \underline{Z}_T \\ X = \text{limit in Bagchi's Th.} \end{cases}$

Then the corollary follows from Bagchi's Th. and:

<u>Prop.</u> $\mathrm{Supp}(X) = \mathcal{H}(\mathbb{D})^\times \cup \{0\}$.