29.4.2021

Proof of lemma (B.3.3) [X_n, X are all M-valued]

Recall : $x \in \mathrm{Supp}(X)$

$$\Updownarrow$$

$\forall V \subset X$, open neighb of x,

we have $\mathbb{P}(X \in V) > 0$.

A consequence of convergence in law is that

$$\liminf_{n \to \infty} \mathbb{P}_n(X_n \in V) \geq \mathbb{P}(X \in V) > 0.$$

$\square$

What happens for $\zeta(\frac{1}{2} + it)$, $t \in \Omega_T$?

One can show that

$$\left\{ \begin{array}{l} t \longmapsto \zeta(\frac{1}{2} + it) \\ \Omega_T \longrightarrow \mathbb{C} \end{array} \right.$$

does not converge in law.

Theorem (4.1.2) [Selberg 1940's]

Define real-valued r.v. on Ω_T by

$$\underline{S}_T(t) = \begin{cases} \log|\zeta(\tfrac{1}{2}+it)|, & \zeta(\tfrac{1}{2}+it) \neq 0 \\ 0, & \zeta(\tfrac{1}{2}+it) = 0 \end{cases}$$

only happens (at most) countably many times

Then $\dfrac{\underline{S}_T}{\sqrt{\frac{1}{2}\log\log T}} \xrightarrow[T\to\infty]{\text{law}} \mathcal{N}(0,1)$

standard gaussian

This is much deeper and much harder than Bagchi's Th. !

(Notes : proof by Soundararajan and Radziwiłł)

The renormalization is an issue in this result because it hides any special feature of $\zeta(s)$.
We will see that the limit in Bagchi's Th. has some arithmetic content.

2 – Motivation for Bagchi's Theorem

Write

$$\zeta(s+it) = \sum_{n\geq 1} \frac{1}{n^{s+it}} \qquad (\mathrm{Re}\, s > 1)$$

$$= \prod_p \frac{1}{1 - \frac{1}{p^{s+it}}} \qquad (\mathrm{Re}\, s > 1)$$

So (in this range) $\zeta(s+it)$, for $t \in \Omega_T$, is a function of the (infinite) vector $(p^{-it})_{p \text{ prime}}$.
The probabilistic link is given by:

<u>Proposition</u> (3.2.5) The r.v.

compact topological group

$$\underline{P}_T \begin{cases} \Omega_T \longrightarrow \prod_p \mathbb{S}^1 = \widehat{\mathbb{S}^1} \\ t \longmapsto (p^{-it})_{p \text{ prime}} \end{cases}$$

converge in law as $T \longrightarrow \infty$ to a sequence $(X_p)_p$ of i.i.d uniform on the unit circle.

[<u>Note:</u> this means that (X_p) is uniformly distributed on $\widehat{\mathbb{S}^1}$, i.e. the law of (X_p) is the Haar measure on $\widehat{\mathbb{S}^1}$.]

Now write

$$\zeta(s+it) = \prod_p \frac{1}{1-p^{-s}p^{-it}}$$

$$= f(\underline{P}_T(t))$$

where

$$f : \widehat{\mathbb{S}^1} \longrightarrow \mathbb{C}$$

$$(x_p) \longmapsto \prod_p (1 - p^{-s} x_p)^{-1}$$

s fixed

which is continuous [for $\mathrm{Re}(s) > 1$],

so Prop. + Composition Principle

imply that

$$(\mathrm{Re}\, s > 1) \qquad \boxed{\underset{\underset{\Omega_T}{\Uparrow}}{\zeta(s+it)} \xrightarrow[T \to \infty]{\text{law}} \prod_p \frac{1}{1 - p^{-s} X_p}}$$

(where (X_p) i.i.d uniform on $\mathbb{S}^1$).

Now one observes that the RHS

$$\prod_p \frac{1}{1 - p^{-s} X_p}$$

actually converges almost surely for s fixed with $\boxed{\mathrm{Re}(s) > \frac{1}{2}}$.

Indeed, this is true (actually proved by H. Bohr – Jessen in

1920's – 1930's), and in fact the RHS makes sense as a random function of $s \in \overline{D_r}$, and so the precise version of Bagchi's Th is:

Th. (Bagchi, 1981)

As $T \to \infty$, the r.v. $\underline{Z}_T$ ($\mathcal{H}(D)$-valued) converge in law to the random function

$$Z(s) = \prod_p (1 - p^{-s} X_p)^{-1}$$
$$\uparrow$$
$$\overline{D}$$

with (X_p) i.i.d uniform on $\mathbb{S}^1$.

<u>Note</u>: the product over primes is what remains of arithmetic

nature in $Z(s)$ (so computing the support is not just a question of analysis).

<u>Proof of Prop. 3.2.5</u>

$$\underline{P}_T(t) = (p^{-it})_p$$

(<u>Goal:</u>
$$\underline{P}_T \xrightarrow[T\to\infty]{\text{law}} (X_p)_p$$
)

Since $\widehat{\mathbb{S}^1}$ is a compact abelian group with the product topology, we can use the Weyl Criterion [Prop. B.6.3], which tells us that it is sufficient to prove that

$$\mathbb{E}_T(\chi(\underline{P}_T)) \xrightarrow[T\to\infty]{} 0$$

for any non-trivial character

$$\widehat{\mathbb{S}^1} \longrightarrow \mathbb{S}^1.$$

<u>Lemma</u> — Any character $\chi : \widehat{\mathbb{S}^1} \longrightarrow \mathbb{S}^1$ is of the form

$$\chi((x_p)) = \prod_{p \in S} x_p^{m_p}$$

for some <u>finite</u> set S and some integers $m_p \in \mathbb{Z} - \{0\}$. The trivial character corresponds to $S = \emptyset$.

Assume that this holds. Then

$$\mathbb{E}_T\left(\chi\left(\underline{p}_T\right)\right) = \frac{1}{2T}\int_{-T}^{T} \prod_{p \in S} p^{-it m_p}\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} q^{-it}\, dt$$

where $q = \prod_{p \in S} p^{m_p} \in \mathbb{Q}^\times - \{1\}$, $q > 0$ [because $\chi \neq 1$].

So the integral is

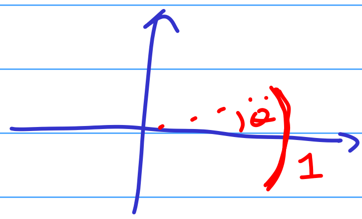
$$\frac{1}{2T}\,\frac{\overbrace{q^{-iT}-q^{iT}}^{\leq 2}}{i\log q} \xrightarrow[T\to\infty]{} 0$$

as we wanted. $\square$

<u>Proof of the lemma</u>:

Let $\chi : \widehat{\mathbb{S}^1} \longrightarrow \mathbb{S}^1$ be a character.

By continuity of χ at 1, there exists an open set $U \subset \widehat{\mathbb{S}^1}$, $1 \in U$, such that

$\chi(U) \subset$ 

$\theta \leq \frac{\pi}{4}$

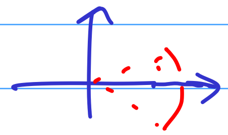
By definition of the product topology, there exists a finite set S of primes and an open neighborhood V of $\underline{1}$ in $\prod_{p\in S} \mathbb{S}^1$

such that

$$U \supset \underset{1\in}{V} \times \underbrace{\prod_{p\notin S} \mathbb{S}^1}_{\text{subgroup of } \widehat{\mathbb{S}_1}}$$

(all coordinates outside S are unrestricted).

Since χ is a group homomorphism

$$\chi\Big(\prod_{p\notin S} \mathbb{S}^1\Big) \subset \chi(U) \subset$$ .

and is a subgroup of this "picture".

However this small arc only contains the subgroup $\{1\}$. So

$$\prod_{p\notin S} \mathbb{S}^1 \subset \operatorname{Ker}(\chi),$$

so there is a character

$$\tilde{\chi} : \prod_{p\in S} \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

such that

$$\chi\big((x_p)\big) = \tilde{\chi}\big((x_p)_{p\in S}\big).$$

But then we have a <u>finite</u> product and

$$\tilde{\chi}\left(\left(x_p\right)_{p\in S}\right) = \tilde{\chi}\left(\prod_{p\in S} x_p\right)$$

viewed as $(\tilde{x}_q)_{q\in S}$

by $\begin{cases} \tilde{x}_q = 1 & \text{if } q \neq p \\ \tilde{x}_p = \underline{1} \end{cases}$

$$= \prod_{p\in S} \tilde{\chi}(x_p)$$

and $\tilde{\chi}$ restricted to the p-component is just a character of S^1, and so it is of the form

$$\tilde{\chi}(x_p) = x_p^{m_p}$$

for some $m_p \in \mathbb{Z}$ [this is because $\begin{cases} S^1 \xrightarrow{\sim} \mathbb{R}/\mathbb{Z} \\ e^{2i\pi\theta} \longleftarrow \theta \end{cases}$ and the characters of $\mathbb{R}/\mathbb{Z}$ are the functions $\theta \longmapsto e(m\theta) = e^{2i\pi m\theta}$ for some $m \in \mathbb{Z}$].

So we conclude that

$$\chi\big(\underset{\substack{\cap \\ \widehat{S^1}}}{(x_p)}\big) = \prod_{p \in S} x_p^{m_p}$$

and we can assume $m_p \neq 0$ by removing from S the primes where $m_p = 0$. For χ non-trivial, we must have some $m_p \neq 0$ so we end up with $S \neq \emptyset$.

$\square$

<u>Remark.</u> More generally, if I is an arbitrary set, $(G_i)_{i \in I}$ are compact abelian groups, the same

argument shows that any character of $G = \prod_{i \in I} G_i$ is of the form

$$\chi((x_i)) = \prod_{i \in S} \chi_i(x_i)$$

for some finite subset $S \subset I$ and some characters

$$\chi_i : G_i \longrightarrow \mathbb{S}^1.$$

Outline of the proof of Bagchi's Theorem

Step 1. Show that the random function $Z(s)$ exists as an $\boxed{\mathcal{H}(D)}$-valued random variable. (Note that this is more delicate than a single s).

[Probability + analysis]

Step 2. Find convenient formulas to represent $\zeta(s+it)$ and $Z(s)$ as series for $\boxed{s \in D}$ [Analysis] ("explicit analytic continuation")

Step 3. Bounds on $\zeta(s+it)$ and $Z(s)$ in certain regions

[Number theory + analysis / probability]

Step 4. Implement convergence in law using test functions

$$f : \mathcal{H}(D) \longrightarrow \mathbb{C}$$

to prove

$$\lim_{T \to \infty} \mathbb{E}_T(f(Z_T)) = \mathbb{E}(f(Z)),$$

assuming that f is Lipschitz-continuous.

[<u>Note:</u> for a fixed $s \in \mathbb{C}$ [Bohr-Jessen] one could try to use finite-dimensional methods, e.g. the method of moments].

3_ <u>Step 1: existence of $Z(s)$</u>

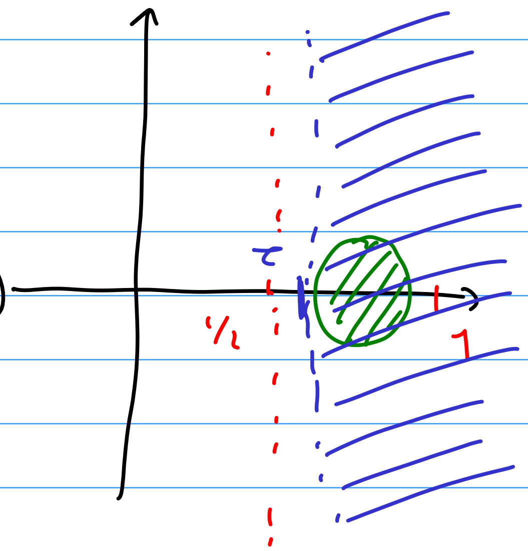
<u>Proposition</u>_ (3.2.9) (X_p) i.i.d. uniform on $\mathbb{S}^1$

Let $\tau \in \left]\tfrac{1}{2}, 1\right[$, and

$$U_\tau = \{ s \in \mathbb{C} \mid \mathrm{Re}(s) > \tau \}$$

(1) Almost surely, the product

$$\prod_p (1 - X_p p^{-s})^{-1} = Z(s)$$



converges for all $s \in U_\tau$, and for any compact set $K \subset U_\tau$ [eg $K = \bar{D}$ if τ small enough]

The values $Z(s)$ for $s \in K$ define an $\mathcal{H}(K)$-valued r.v.

continuous on K, hol. in $\mathring{K}$

(2) Define X_n for $n \geq 1$ by

$$X_n = \prod_{p^\nu \| n} X_p^\nu$$

[so $X_n X_m = X_{nm}$ for all $n, m \geq 1$]

Almost surely the series

$$\sum_{n \geq 1} \frac{X_n}{n^s}$$

converges for all $s \in U_\tau$ and defines for $s \in K \subset U_\tau$ (compact) an $\mathcal{H}(K)$-valued r.v., which is (almost surely) equal to $Z(s)$.