

Student Seminar: Monstrous Moonshine

ETH Zürich, Frühjahrssemester 2021

Organizer: Tim-Henrik Buelles, HG J 16.4, buelles@math.ethz.ch

Time and Location: Fri 12 - 14, Zoom

Goal: The goal of this seminar is to understand the equation

$$196884 = 196883 + 1 \tag{1}$$

The left-hand side of (1) is the first non-trivial coefficient of the modular invariant j -function. On the right hand-side are the dimensions of the smallest irreducible representations of the largest sporadic simple group, the Monster group \mathbb{M} .

Content: The talks will give an overview over the topics necessary to understand (1), namely modular forms, lattices, the classification of finite simple groups, and the theory of Vertex Operator Algebras. The goal is not to give a complete and detailed understanding of those topics, but to provide enough information to understand what (1) means, why it is surprising, and to give a general idea of how this mysterious relation can be explained.

Prerequisites: Algebra I and II. Some familiarity with modular forms and Lie algebras is helpful, but not crucial: all necessary concepts will be introduced in the early talks.

Literature: This seminar will be heavily based on Terry Gannon's book [1]. For a briefer overview see also [2] and part 2 of [3]. The references assigned to each talk cover the bare minimum. You are very much encouraged to find additional references and explore (and share with us) connections to other subjects as well.

Organization: One week before the talk, you should discuss with the organizer (e.g. following the Fri 12 - 14 session). The purpose is to discuss the objective and rough schedule of your talk. It is not meant to go into details or technicalities. Since the topic of this seminar is rather advanced, it should be a good idea to share your questions with the other participants and to discuss with each other.

Schedule of Talks

Talks marked with * are somewhat more advanced.

1 Modular Forms

Speaker: Adrian Spiess

Topics: Basic Definitions, Eisenstein series and Klein j -function

References: [4, Section 1-2]

Date: 05 March

2 Lattices I

Speaker: Ruilong Wäckerlin

Topics: Basic definitions: Integral lattices, even lattices, dual lattices, unimodular lattices, Theta functions. Poisson Resummation. Lattice theta functions. Hecke/Schoenberg.

References: [5, Chapters 1-2], [4, Section 3]

Date: 12 March

3 Lattices II

Speaker: Lukas Oestmann

Topics: Gluing constructions. Niemeier Lattices in $d = 24$. Golay Code. Leech lattice

References: [5, Chapters 4, 16], [6, Chapters 2, 3]

Date: 19 March

4 Group Theory

Speaker: Julian Huber, Nadja Häusermann

Topics: *1st part:* Normal subgroups, Simple Groups, Jordan-Hölder Theorem

2nd part: Representations of finite groups, Character formulae.

References: [7, Chapters I.3] resp. [7, XVIII.1 – XVIII.5]

Date: 26 March

5 Lie Theory I

Speaker: Reto Kaufmann

Topics: Lie algebras. Jacobi identity. Classification. Highest weight representations.

References: [2, Sections 1.4, 1.5], [8, Sections 13.1, 13.2], [9]

Date: 16 April

6 Lie Theory II*

Speaker: Silvio Barandun

Topics: Affine Lie Algebras. The Virasoro Algebra.

References: [1, Section 3], [8, Section 14]

Date: 23 April

7 Classification of simple groups

Speaker: Gaspard Mudry

Topics: Give an overview over the infinite families and the sporadic groups

References: [10, Sections 2–5]

Date: 30 April

8 The Monster Group*

Speaker: Nico Ehrler

Topics: Construction of the Monster. Representations.

References: [5, Chapter 29], [10, Section 5.8]

Date: 07 May

9 Vertex Operator Algebras I*

Speaker: Ole Spjeldnæs

Topics: Basic definitions: Formal power series. The δ series. Binomial convention. Axioms of VOA. Weak Commutativity. Correlation functions.

References: [1, 5.1 and 5.3.1], [11, Section 2], [12, Chapters 1–3]

Date: 14 May

10 Vertex Operator Algebras II*

Speaker: Lukas Bertsch

Topics: Examples: Free boson, Heisenberg algebra. Lattice algebras.

References: [1, 5.2.2], [11, Section 3], [12, Sections 6.3–6.5]

Date: 21 May

11 Vertex Operator Algebras III*

Speaker: Leon Staresinic

Topics: Modules of VOAs. Rational CFTs. Self-dual CFTs. Zhu's Algebra. Modular invariance.

References: [1, 5.3], [13]

Date: 28 May

12 The Monster VOA*

Speaker: Malcolm Cameron

Topics: \mathbb{Z}_2 orbifolds. FLM construction.

References: [1, 4.3.4, 5.3.6, 7.2]

Date: 04 June

References

- [1] Terry Gannon. *Moonshine beyond the Monster*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2006. The bridge connecting algebra, modular forms and physics.
- [2] Terry Gannon. Monstrous moonshine: the first twenty-five years. *Bull. London Math. Soc.*, 38(1):1–33, 2006.
- [3] Terry Gannon. Monstrous Moonshine and the classification of CFT. In *Conformal field theory (Istanbul, 1998)*, volume 102 of *Front. Phys.*, page 66. Adv. Book Program, Perseus Publ., Cambridge, MA, 2000.
- [4] Don Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008.
- [5] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.
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- [11] Chongying Dong. Introduction to vertex operator algebras. I. *Sūrikaiseikikenkyūsho Kōkyūroku*, pages 1–25, 1995. Moonshine and vertex operator algebra (Japanese) (Kyoto, 1994).
- [12] James Lepowsky and Haisheng Li. *Introduction to vertex operator algebras and their representations*, volume 227 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [13] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.