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Symmetric Spaces | Spring 2021

Exercise Sheet 1

In this exercise sheet, we consider the group $G = \mathrm{SL}_n(\mathbb{R})$. Our aim is to prove that $G_\mathbb{Z} = \mathrm{SL}_n(\mathbb{Z})$ is a lattice in G.

1) Argue that $G_\mathbb{Z}$ is discrete in G and that both G and $G_\mathbb{Z}$ are unimodular.

From this we know that $G/G_\mathbb{Z}$ admits a nonzero G-invariant measure μ which is unique up to a non-zero constant. In order to show that $G_\mathbb{Z}$ is a lattice we have to show that $\mu(G/G_\mathbb{Z}) < \infty$. For this, we use the following fact:

2) Assume that there exists a measurable set $A \subseteq G$ of finite measure such that every $G_\mathbb{Z}$-orbit intersects A, i.e. for every $g \in G$ there exists some $\gamma \in G_\mathbb{Z}$ such that $g\gamma \in A$. Show that $\mu(G/G_\mathbb{Z})$ is finite.

We delay the general proof for a moment to consider a classical case, namely $n = 2$. It is also closely related to symmetric spaces. In fact, the complex upper half plane $\mathcal{H}$ is a globally symmetric space, as we will see. As a Riemannian manifold it is isomorphic to the hyperbolic plane H^2, a symmetric space of non-compact type. It is even a complex manifold and the complex structure is compatible with its structure as a Riemannian manifold. Thus, it belongs to the important subclass of Hermitian symmetric spaces.

3) In this exercise, set $G = \mathrm{SL}_2(\mathbb{R})$ and $G_\mathbb{Z} = \mathrm{SL}_2(\mathbb{Z})$.

(a) Show that the map sending

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ to } z \longmapsto g \cdot z := \frac{az+b}{cz+d}$$

is a group homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$, where $\mathrm{Bih}(\mathcal{H})$ denotes the biholomorphic maps of the complex upper half plane $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. Show that its kernel is $\{\pm I\}$ where I denotes as usual the 2×2 identity matrix.

(b) Prove that the induced homomorphism

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\} \to \mathrm{Bih}(\mathcal{H})$$

of a) is actually an isomorphism. For the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ from above determine the orbit Gi and stabilizer K of $i \in \mathcal{H}$. (Show also that K is compact.) Using this, show that we have a diffeomorphism

$$G/K \longrightarrow \mathcal{H},\ g \longmapsto g \cdot i.$$

(c) Set $K = \mathrm{SO}_2(\mathbb{R})$,

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R}, a > 0 \right\},$$

$$A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \mid y \in \mathbb{R}^+ \right\}, \text{ and}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

Prove the Iwasawa decomposition, i.e. show that

$$P \times K \longrightarrow G, \ (p, k) \longmapsto pk$$

and

$$N \times A \longrightarrow P, \ (n, a) \longmapsto na$$

are diffeomorphisms. Are these also Lie group isomorphisms? Show that P is a semidirect product $N \rtimes A$ and that we have the diffeomorphism $N \times A \cong \mathcal{H}$.

(d) Prove that K is unimodular by showing that $d\mu_{\mathcal{H}} = y^{-2} dx dy$, $z = x + iy$, is a G-invariant volume form on the G-homogeneous space $\mathcal{H}$.

(e) Show that $\mathcal{F} = \{z \in \mathcal{H} \mid (|z| > 1 \text{ and } -1/2 \leq \mathrm{Re}(z) < 1/2) \text{ or } (|z| = 1 \text{ and } -1/2 \leq \mathrm{Re}(z) \leq 0)\}$ is a fundamental domain for the action of $G_{\mathbb{Z}}$ on $\mathcal{H}$.

<u>Hint:</u> For every $G_{\mathbb{Z}}$-orbit $G_{\mathbb{Z}} z$, $z \in \mathcal{H}$, consider $w \in G_{\mathbb{Z}} z$ with maximal imaginary part.

(f) Show that the volume of $\mathcal{F}$ with respect to $\mu_{\mathcal{H}}$ is $\pi/3$. Deduce that $\mu(G/G_{\mathbb{Z}}) < \infty$.

(g) ($\star$) G/K looks like a (Riemannian globally) symmetric space. Give the geodesic symmetry s_i at i by using the formula we saw (but have not proved yet) in the lecture.

(h) ($\star$) If you are courageous enough, deduce in 3f) that (normalizations[1] as in Exercise 4 below)

$$\mu(G/G_{\mathbb{Z}}) = \frac{\pi^2}{6} (= \zeta(2), \text{ where } \zeta(z) \text{ is the Riemann zeta function}).$$

A nice formula, isn't it?

<u>Hint:</u> Be careful, the measure $\mu_{\mathcal{H}}$ is not what one gets "group-theoretically" by using the Haar measures μ_G and μ_K with their standard normalizations as below (, which you – as everyone else – should use).

[1] Here, if G is a locally compact groups and H a closed subgroup of G (both unimodular), then the (up to constant) unique non-zero semi-invariant measure on G/H is normalized by

$$\int_G f(g) d\mu_G(g) = \int_{G/H} \left(\int_H f(gh) d\mu_H(h) \right) d\mu_{G/H}(gH).$$

Now, let us come back to the general case. First of all, we have to find a (bi-invariant) Haar measure on μ_G, which is not as easy as it is for $\mathrm{GL}_n(\mathbb{R})$, where we can just write down a volume form. Last semester, we gave a Haar measure for $\mathrm{SL}_2(\mathbb{R})$ by decomposing it as a measure on the upper half plane $\mathcal{H}$ and on $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, the above exercise repeats parts of our proof there. In the general case, more refined tools are necessary, such as

4) **The Iwasawa decomposition of** $\mathrm{SL}_n(\mathbb{R})$**.** In generalization of the groups in Exercise 3 above, we consider here the following Lie subgroups of $G = SL_n(\mathbb{R})$:

$$K = \mathrm{SO}_n(\mathbb{R}), \text{ the special orthogonal group,}$$

$$P = \left\{ (a_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathrm{SL}_n(\mathbb{R}) \mid a_{ij} = 0 \text{ if } i > j,\, a_{ii} > 0 \text{ for all } 1 \le i \le n \right\},$$

$$A = \left\{ (a_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathrm{SL}_n(\mathbb{R}) \mid a_{ij} = 0 \text{ if } i \ne j,\, a_{ii} > 0 \text{ for all } 1 \le i \le n \right\}, \text{ and}$$

$$N = \left\{ (n_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathrm{SL}_n(\mathbb{R}) \mid n_{ij} = 0 \text{ if } i > j,\, n_{ii} = 1 \text{ for all } 1 \le i \le n \right\}.$$

(a) Show (again) that

$$K \times P \longrightarrow G,\ (k,p) \longmapsto kp$$

and

$$A \times N \longrightarrow P,\ (a,n) \longmapsto an$$

are diffeomorphisms.

Hint: For a given $g \in G$, there exists a unique orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ such that $ge_i = \alpha_{i1} v_1 + \dots + \alpha_{ii} v_i$, $\alpha_{ii} > 0$, where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$.

Show that P is a semidirect product $A \ltimes N$.

(b) Are the above diffeomorphisms group homomorphisms?

5) **A construction of a Haar measure on** $\mathrm{SL}_n(\mathbb{R})$**.**

(a) Show that $A \cong (\mathbb{R}^{>0})^{n-1}$ (as real Lie groups). Show that the volume form $x^{-1}dx$ on $\mathbb{R}^{>0}$ gives a bi-invariant Haar measure. Write down the resulting bi-invariant Haar measure μ_A on A.

(b) There exists a canonical diffeomorphism $N \cong \mathbb{R}^{(n-1)(n-2)/2}$ by mapping a matrix the coefficients of its strictly upper triangle. Show that the Lebesque volume form $\prod_{i<j} dx_{ij}$ on $\mathbb{R}^{(n-1)(n-2)/2}$ gives a bi-invariant Haar measure on N. (Hint: From elementary linear algebra, you know that every matrix in N decomposes as a product of elementary matrices.)

(c) Show that given locally compact groups A, N with Haar measures μ_A and μ_N a right Haar measure ν_P on $P = A \ltimes N$ is given by $d\nu_P(a,n) =$

$\mathrm{mod}_N(a)d\mu_A(a)d\mu_N(n)$, where $\mathrm{mod}_N(a)$ is the modulus of the automorphism $n \mapsto ana^{-1}$ with respect to μ_N.

(d) Let μ_K be a bi-invariant Haar measure on K, normalized such that $\mu_K(K) = 1$. Show that all non-zero positive Radon measures on G which are right-invariant for K and left-invariant for P must be (bi-invariant) Haar measures of G. Use this to show that

$$d\mu_K(k)d\nu_P(p) = \rho(a)d\mu_K(k)d\mu_A(a)d\mu_N(n),\ \rho(a) = \prod_{1\le i<j\le n}\frac{a_{ii}}{a_{jj}}$$

gives a (bi-invariant) Haar measure on G.

6) Siegel sets[2] **in** $\mathrm{SL}_n(\mathbb{R})$: For every positive $t,u > 0$ we set

$$A_t = \left\{(a_{ij})_{\substack{1\le i\le n\\1\le j\le n}} \in A \mid a_{ii} \le t a_{(i+1)(i+1)} \text{ for all } 1 \le i \le (n-1)\right\}, \text{ and}$$

$$N_u = \left\{(n_{ij})_{\substack{1\le i\le n\\1\le j\le n}} \in N \mid |n_{ij}| \le u\right\}.$$

Every set $\mathcal{S}_{t,u} = KA_tN_u$ is called a Siegel set.

(a) $\mathcal{S}_{t,u}$ has finite measure for all $t,u > 0$.

(b) Show that $N = N_{1/2}N_{\mathbb{Z}}$, where $N_{\mathbb{Z}} = \left\{(n_{ij})_{\substack{1\le i\le n\\1\le j\le n}} \in N \mid N_{ij} \in \mathbb{Z}\right\}$.

(c) Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$ and $e_1 = (1,0,\ldots,0)$. Argue that

$$\Phi : G \longrightarrow \mathbb{R}^\times,\ g \mapsto \|ge_1\|$$

is a continuous function on G.

(d) The function Φ attains a positive minimum on each $G_{\mathbb{Z}}$-orbit $gG_{\mathbb{Z}}$ in G. Show that this minimum must be attained at a point $gG_{\mathbb{Z}} \cap \mathcal{S}_{2/\sqrt{3},1/2}$.
<u>Hint:</u> $\Phi(kan) = a_{11}$, where $g = kan$ is the Iwasawa decomposition. It can attain these minima only for points $g \in G$ satisfying $a_{11} \le (2/\sqrt{3})a_{22}$. Use this fact in an induction on n, the case $n = 1$ being clear (why?).

(e) Conclude that $G_{\mathbb{Z}}$ is a lattice.

[2] named after Carl Ludwig Siegel (1896-1981), a number theorist, who pioneered the theory of automorphic forms of several variables. He also calculated $\mu(G/G_{\mathbb{Z}})$ explicitly for every natural n. In fact, he showed (with our Haar measure μ_G) that

$$\mu(G/G_{\mathbb{Z}}) = \zeta(2)\zeta(3)\ldots\zeta(n).$$

This can be done since $\mathcal{S}_{2/\sqrt{3},1/2}$ is not too far from being a fundamental domain for $G_{\mathbb{Z}}$. Finally, it is noteworthy that Siegel did not compute $\mu(G/G_{\mathbb{Z}})$ just for fun, as you might do nowadays, but in order to get information on the asymptotic growth of the number of certain lattice points.

Due date: Thursday, 11/03/2021

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

`solution_<number exercise sheet>_<last name>_<first name>.pdf`

For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

`solution_2_Miller_Alice.pdf`

Solutions that fail to comply with the above requirements will be ignored.