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Symmetric Spaces — Spring 2021

# Exercise Sheet 2

**Exercise 1.(Invariant Riemannian metrics on homogeneous spaces):**

In the first exercise class we saw that every homogeneous $G$-manifold $M$ is diffeomorphic to a quotient $G/H$, where $H = G_p < G$ is the stabilizer subgroup of a point $p \in M$. The diffeomorphism $F\colon G/H \to M$ is given by $F(gH) = g \cdot p$. Moreover, we saw that the set $R(M)^G$ of $G$-invariant Riemannian metrics on $M$ can be identified with the set $\mathrm{Sym}_+(T_pM)^H$ of $H$-invariant inner products on the tangent space $T_pM$.

Complete our discussion by showing the following:

a) Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of $G$ and $H$, respectively. Then the differential $dF_e\colon \mathfrak{g}/\mathfrak{h} \cong T_eG/H \to T_pM$ induces a bijection between $H$-invariant inner products on $T_pM$ and $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.

b) Show that every $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle \in \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})$ is also $\mathrm{ad}(\mathfrak{h})$-invariant, i.e.

$$\langle \mathrm{ad}(X)Y, Z\rangle + \langle Y, \mathrm{ad}(X)Z\rangle = 0$$

for all $X \in \mathfrak{h}$, $Y, Z \in \mathfrak{g}/\mathfrak{h}$.

If $H$ is connected, the converse holds as well: Every $\mathrm{ad}(\mathfrak{h})$-invariant inner product is $\mathrm{Ad}(H)$-invariant.

c) Let $G = \mathrm{GL}(n,\mathbb{R})$ and let $d_1,\dots,d_m \in \mathbb{N}$ such that $d_1 + \cdots + d_m = n$. Denote by $P < G$ the subgroup that consists of block upper triangular matrices of the form

$$\begin{pmatrix} \boxed{B_1} & & * \\ & \ddots & \\ 0 & & \boxed{B_m} \end{pmatrix},$$

where $B_i \in \mathrm{GL}(d_i,\mathbb{R})$, $i = 1,\dots,m$.

Use the above characterization to show that there are no $G$-invariant Riemannian metrics on $G/P$.

<u>Remark:</u> The quotient space $G/P$ can be interpreted as the flag variety of partial flags $\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = \mathbb{R}^n$, where $\dim V_i = d_1 + \cdots + d_i$, $i = 1,\dots,m$.

### Exercise 2.(Compact Lie groups as symmetric spaces):

Let $G$ be a compact connected Lie group and let

$$G^* = \{(g,g) \in G \times G : g \in G\} < G$$

denote the diagonal subgroup.

a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to $G$.

b) Using the above, explain how any compact connected Lie group $G$ can be regarded as a Riemannian globally symmetric space.

c) Let $\mathfrak{g}$ denote the Lie algebra of $G$. Show that the exponential map from $\mathfrak{g}$ into the Lie group $G$ coincides with the exponential map from $\mathfrak{g}$ into the Riemannian *globally symmetric space $G$*.

### Exercise 3.(Compact semisimple Lie groups as symmetric spaces):

A compact semisimple Lie group $G$ has a bi-invariant Riemannian structure $Q$ such that $Q_e$ is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. If $G$ is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure $Q^*$ from the Killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

### Exercise 4.(Constant sectional curvature determines isometry type):

Show or look up the following theorem:

*Let $M$ be a complete and simply connected Riemannian manifold of dimension $n$ and constant sectional curvature $K$. Then $M$ is isometric to:*

*a) the hyperbolic $n$-space $\mathbb{H}^n$, if $K \equiv -1$;*

*b) the Euclidean $n$-space $\mathbb{R}^n$, if $K \equiv 0$;*

*c) the $n$-sphere $\mathbb{S}^n$, if $K \equiv 1$.*

### Exercise 5.(Closed differential forms):

Let $M$ be a Riemannian globally symmetric space and let $\omega$ be a differential form on $M$ invariant under $\mathrm{Isom}(M)^\circ$. Prove that $d\omega = 0$.

**Due date:** Thursday, 25/03/2021

**Please, upload your solution via the SAM upload tool.**

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

`solution_<number exercise sheet>_<last name>_<first name>.pdf`

**For example:** If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

`solution_2_Miller_Alice.pdf`

**Solutions that fail to comply with the above requirements will be ignored.**