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Symmetric Spaces

Spring 2021

Exercise Sheet 4

Exercise 1.(Complexification and Killing form):

Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let K_0, K and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}, \mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

a) $K_0(X,Y) = K(X,Y)$ for all $X, Y \in \mathfrak{l}_0$;

b) $K^{\mathbb{R}}(X,Y) = 2 \cdot \mathrm{Re}(K(X,Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.

Exercise 2.(Semisimple OSLAs):

Let $(\mathfrak{l}, \Theta)$ be an orthogonal symmetric Lie algebra with $\mathfrak{l}$ semisimple. Show that:

a) $\mathfrak{u}$ equals its normalizer in $\mathfrak{l}$;

b) if $\mathfrak{u}$ contains no ideal in $\mathfrak{l}$ then $[\mathfrak{e}, \mathfrak{e}] = \mathfrak{u}$.

Exercise 3.($\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$):

Exhibit an explicit isomorphism between $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$.

<u>Hint:</u> Consider the the vector space V of 2×2-skew-Hermitian matrices and endow it with the quadratic form $q(v) := \det(v)$. Now, let $\mathrm{SL}(2,\mathbb{C})$ act on V via $g.v := gv\bar{g}^t$.

Exercise 4.(Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$):

Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

Due date: Thursday, 29/04/2021

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

`solution_<number exercise sheet>_<last name>_<first name>.pdf`

For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

`solution_2_Miller_Alice.pdf`

Solutions that fail to comply with the above requirements will be ignored.