

EXERCISE SHEET 5

Exercise 1. (Maximal abelian subspaces and regular elements in $\mathfrak{sl}(n, \mathbb{R})$):

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. A Cartan decomposition of \mathfrak{g} is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p} = \{X \in \mathfrak{sl}(n, \mathbb{R}) : X = X^t\}$ and $\mathfrak{k} = \{X \in \mathfrak{sl}(n, \mathbb{R}) : X = -X^t\}$. Define

$$\mathfrak{a} = \left\{ \text{diag}(t_1, \dots, t_n) : t_j \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \right\}.$$

- Prove that \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} .
- Prove (without appealing to the general theorem) any maximal abelian subspace of \mathfrak{p} is of the form $S\mathfrak{a}S^{-1}$ where $S \in SO(n)$.
- Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct.

Exercise 2. (Maximal abelian subspaces and regular elements in $\mathfrak{sp}(2n, \mathbb{R})$):

Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$. Recall that a Cartan decomposition of \mathfrak{g} is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = A^t, B = B^t \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A = -A^t, B = B^t \right\}.$$

- Define

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \text{diag}(t_1, \dots, t_n) \right\}.$$

Prove that \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} .

- Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct and non-zero.

Exercise 3. (The Siegel upper half space):

Let

$$\mathbb{H}_n := \{Z \in \mathbb{C}^{n \times n} : Z = Z^t, \text{Im}(Z) \text{ is positive-definite}\}.$$

Find an explicit isomorphism between \mathbb{H}_n and $\mathrm{Sp}(2n, \mathbb{R})/(\mathrm{SO}(2n) \cap \mathrm{Sp}(2n, \mathbb{R}))$. Use this and the previous exercise to construct a maximal flat of \mathbb{H}_n .

Hint : Consider the map

$$\begin{aligned} \phi : \mathrm{Sp}(2n, \mathbb{R}) &\rightarrow \mathbb{H}_n, \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &\mapsto (Ai + B) \cdot (Ci + D)^{-1}. \end{aligned}$$

Exercise 4. (Irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$):

Let $V = \mathbb{C}[X, Y]$ be the vector space of polynomials in two variables. Let V_m denote the vector subspace of all homogeneous polynomials of degree m . This has a basis given by the monomials $X^m, X^{m-1}Y, \dots, Y^m$. We turn this vector subspace into a module for $\mathfrak{sl}(2, \mathbb{C})$ by defining a Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(V_m)$ in the following way

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X \frac{\partial}{\partial Y}, \quad \phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = Y \frac{\partial}{\partial X}, \quad \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}.$$

Show that this defines an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$.

Due date: Thursday, 20/05/2021

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

`solution_<number exercise sheet>_<last name>_<first name>.pdf`

For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

`solution_2.Miller_Alice.pdf`

Solutions that fail to comply with the above requirements will be ignored.