In this exercise sheet, we consider the group $G = SL_n(\mathbb{R})$. Our aim is to prove that $G_{\mathbb{Z}} = SL_n(\mathbb{Z})$ is a lattice in G.

1. Argue that $G_{\mathbb{Z}}$ is discrete in G and that both G and $G_{\mathbb{Z}}$ are unimodular.

Solution: Since $SL_n(\mathbb{Z})$ is contained in the topological space $\mathbb{Z}^{n\times n}$, a discrete set of $\mathbb{R}^{n\times n}$, the fact that $G_{\mathbb{Z}}$ is discrete in G follows by restriction from $\mathbb{R}^{n\times n}$ to G. As a discrete group $G_{\mathbb{Z}}$ is unimodular (note that each counting measure on $G_{\mathbb{Z}}$ is a bi-invariant Haar measure). G is unimodular as a closed normal Lie subgroup (Lie Groups I, Proposition 1.5) of the unimodular Lie group $GL_n(\mathbb{R})$. Recall that $GL_n(\mathbb{R})$ has bi-invariant Haar measure $|\det(x_{ij})|^{-1}dx_{11}\dots dx_{nn}$.

From this we know that $G/G_{\mathbb{Z}}$ admits a nonzero G-invariant measure μ which is unique up to multiplication by a non-zero constant (Lie Groups I, Theorem 1.9). In order to show that $G_{\mathbb{Z}}$ is a lattice we have to prove that $\mu(G/G_{\mathbb{Z}}) < \infty$. For this, we use the following fact:

2. Assume that there exists a set $A \subseteq G$ of finite measure such that every $G_{\mathbb{Z}}$-orbit intersects A, i.e. for every $g \in G$ there exists some $\gamma \in G_{\mathbb{Z}}$ such that $g\gamma \in A$. Show that $\mu(G/G_{\mathbb{Z}})$ is finite.

Solution: Weyl's formula (again Lie Groups I, Theorem 1.9) for the (integrable) characteristic function χ_A of A in G states that

$$\mu(A) = \int_G \chi_A(g)dg = \int_{G/G_{\mathbb{Z}}} \left(\int_{G_{\mathbb{Z}}} \chi_A(\gamma h)dh \right) d(\gamma G_{\mathbb{Z}}).$$

By assumption, the inner integral is always greater than some absolute constant depending only on the Haar measure of $G_{\mathbb{Z}}$. (Recall that the Haar measure on $G_{\mathbb{Z}}$ must be a counting measure.) Thus, we infer from the above equation that $\mu(A) \geq c\mu(G/G_{\mathbb{Z}})$.

We delay the general proof for a moment to consider a classical case, namely $N = 2$. It is also closely related to symmetric spaces. In fact, the complex upper half plane $\mathcal{H}$ is a globally symmetric space, as we will see. As a Riemannian manifold it is isomorphic to the hyperbolic plane H^2, a symmetric space of non-compact type. It is even a complex manifold and the complex structure is compatible with its structure as a Riemannian manifold. Thus, it belongs to the important subclass of Hermitian symmetric spaces.

3. In this exercise, set $G = SL_2(\mathbb{R})$ and $G_{\mathbb{Z}} = SL_2(\mathbb{Z})$.

a) Show that the map sending

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \text{ to } \left(z \longmapsto g\cdot z := \frac{az+b}{cz+d} \right)$$

is a homomorphism $SL_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$, where $\mathrm{Bih}(\mathcal{H})$ denotes the biholomorphism group of the complex upper half plane $\mathcal{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. Show that its kernel is $\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$.

Solution: Define the automorphy factor $j : SL_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ by

$$j(\gamma, z) = (cz+d), \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For all $z \in \mathcal{H}$ one has the trivial matrix relation

$$\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}.$$

Given $\alpha, \beta \in SL_2(\mathbb{R})$ one now calculates $\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix}$ in two different ways: This yields both

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha\beta, z) \begin{pmatrix} (\alpha\beta)z \\ 1 \end{pmatrix}$$

and

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha, \beta z) j(\beta, z) \begin{pmatrix} \alpha(\beta z) \\ 1 \end{pmatrix}.$$

One deduces the automorphy relation

$$j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z) \text{ for all } \alpha, \beta \in SL_2(\mathbb{R}),\ z \in \mathcal{H},$$

and furthermore

$$(\alpha\beta)z = \alpha(\beta z) \text{ for } \alpha, \beta \in SL_2(\mathbb{R}),\ z \in \mathcal{H}.$$

This relation shows that the map $SL_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$ is indeed a homomorphism. The remaining assertions are straightforward to verify.

b) Prove that the induced homomorphism

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \longrightarrow \mathrm{Bih}(\mathcal{H})$$

of a) is actually an isomorphism. For the action of $SL_2(\mathbb{R})$ on $\mathcal{H}$ from above determine the orbit Gi and stabilizer K of $i \in \mathcal{H}$. Show also that K is compact. Using this, show that we have a diffeomorphism

$$G/K \longrightarrow \mathcal{H},\ g \longmapsto g \cdot i.$$

Solution: It suffices to show that every biholomorphism of $\mathcal{H}$ is actually in the image of $SL_2(\mathbb{R}) \to \mathrm{Bih}(\mathcal{H})$. The Cayley transform

$$\varphi : z \longmapsto \frac{z-i}{z+i}$$

sends the upper half plane $\mathcal{H}$ biholomorphically onto the unit disc $\mathcal{D}$ around 0. Therefore, it establishes an isomorphism $\mathrm{Bih}(\mathcal{H}) \xrightarrow{\sim} \mathrm{Bih}(\mathcal{D})$ by $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$. Since

$$\left[\frac{1}{\sqrt{2i}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\right] SL_2(\mathbb{R}) \left[\frac{1}{\sqrt{2i}} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\right] = SU_{1,1}(\mathbb{C})$$

with

$$SU_{1,1}(\mathbb{C}) = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1\}$$

it suffices to show that every biholomorphism of $\mathcal{D}$ is in the image of the homomorphism $g \mapsto f_g$ defined as follows: For each

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU_{1,1}(\mathbb{C}),\ a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1,$$

the map

$$f_g : z \longmapsto (az+b)/(\bar{b}z + \bar{a})$$

is a biholomorphism of $\mathcal{D}$. Indeed,

$$\begin{aligned}|(az+b)/(\bar{b}z+\bar{a})|<1 &\iff (az+b)(\bar{a}\bar{z}+\bar{b})<(\bar{b}z+\bar{a})(b\bar{z}+a)\\ &\iff |a|^2|z|^2+|b|^2<|b|^2|z|^2+|a|^2\\ &\iff |z|<1.\end{aligned}$$

shows that f_g sends $\mathcal{D}$ to $\mathcal{D}$. It has inverse $f_{g^{-1}}$ because the argument from (3a) shows that $f_{g_2}\circ f_{g_1}=f_{g_2g_1}$, i.e. $g\mapsto f_g$ is a homomorphism. Now, let φ be an arbitrary biholomorphism of $\mathcal{D}$. Then,

$$\psi=(f_g\circ\varphi),\text{ where } g=\frac{1}{\sqrt{1+|\varphi(0)|^2}}\begin{pmatrix}1 & -\varphi(0)\\ -\overline{\varphi(0)} & 1\end{pmatrix}\in SU_{1,1}(\mathbb{C}),$$

is also a biholomorphism of $\mathcal{D}$ with the additional property that $\psi(0)=0$. The classical Schwarz Lemma yields $\psi(z)=e^{i\theta}z$, $\theta\in[0,2\pi)$. One infers easily $\varphi\in SU_{1,1}(\mathbb{C})$ from this and that $PSL_2(\mathbb{R})\to\mathrm{Bih}(\mathcal{H})$ is an isomorphism.

Since

$$\begin{pmatrix}y^{1/2} & x\\ 0 & y^{-1/2}\end{pmatrix}i=x+iy$$

for any $x\in\mathbb{R}$, $y\in\mathbb{R}^+$ the orbit Gi equals $\mathcal{H}$. Furthermore, given $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ the following equivalences hold

$$gi=i\iff\frac{ai+b}{ci+d}=i\iff ai+b=-c+id.$$

Taking real and imaginary parts, one deduces that $SO_2(\mathbb{R})$ is the stabilizer of i. That $G/K\longrightarrow\mathcal{H}$ is a diffeomorphism follows from standard arguments of differential geometry: In fact, it is a homeomorphism by Helgason, Theorem II.3.2, and a diffeomorphism by Helgason, Theorem II.4.3.(a).

c) Set $K=SO_2(\mathbb{R})$,

$$P=\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}\mid a,b\in\mathbb{R},\ a>0\},$$

$$A=\{\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\mid y\in\mathbb{R}^+\},\text{ and}$$

$$N=\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\mid x\in\mathbb{R}\}.$$

Prove the Iwasawa decomposition, i.e. show that

$$P\times K\longrightarrow G,\ (p,k)\longmapsto pk$$

and

$$N\times A\longrightarrow P,\ (n,a)\longmapsto na$$

are diffeomorphisms. Are these also Lie group isomorphisms? Show that P is a semidirect product $N\rtimes A$ and that we have the diffeomorphism $N\times A\cong\mathcal{H}$.

Solution: Both $P\cap K=\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\}$ and $N\cap A=\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\}$ are easily verifiable identities. The two maps $P\times K\to G$ and $N\times A\to P$ are hence injective. Let us show next that $P\times K\to G$ is surjective: Given $g\in G$ we can regard the matrix of $g=(f_1,f_2)$, $f_i\in\mathbb{R}^2$, as determining a basis $\{f_1,f_2\}$ of $\mathbb{R}^2$. We introduce on $\mathbb{R}^2$ the canonical scalar product $\langle\cdot,\cdot\rangle$. Now, we can use the Gram-Schmidt algorithm for $\langle\cdot,\cdot\rangle$ on $\{f_1,f_2\}$ to find a matrix $p\in P$

such that $p^{-1}g \in K$. In addition, a simple algebraic argument shows that $N \times A \to P$ is surjective. Neither $P \times K \to G$ nor $N \times A \to P$ is a homomorphism. However, since $NA = P$, $N \cap A = \{1\}$ and $N \lhd P$ one has $P = N \rtimes A$.

Caveat: Showing that a differentiable map is bijective does not suffice to prove that it is a diffeomorphism, i.e. has a differentiable inverse. However, the Gram-Schmidt algorithm provides us directly with a differentiable inverse.

Finally, $\mathcal{H} \cong G/K \cong (N \times A \times K)/K \cong N \times A$.

d) Prove that K is unimodular by showing that $d\mu_{\mathcal{H}} = y^{-2}dxdy$, $z = x+iy$, is a G-invariant volume form on the G-homogeneous space $\mathcal{H}$.

Solution: The derivative of $f_g : z \mapsto \frac{az+b}{cz+d}$ is

$$\frac{a(cz+d) - c(az+b)}{(cz+d)^2} = (cz+d)^{-2}.$$

The Cauchy-Riemann differential equations imply

$$\begin{pmatrix} \frac{\partial(\operatorname{Re} f_g)}{\partial x} & \frac{\partial(\operatorname{Re} f_g)}{\partial y} \\ \frac{\partial(\operatorname{Im} f_g)}{\partial x} & \frac{\partial(\operatorname{Im} f_g)}{\partial y} \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(cz+d)^{-2} & -\operatorname{Im}(cz+d)^{-2} \\ \operatorname{Im}(cz+d)^{-2} & \operatorname{Re}(cz+d)^{-2} \end{pmatrix}$$

and the determinant Δ of this matrix is

$$\left[\operatorname{Re}(cz+d)^{-2}\right]^2 + \left[\operatorname{Im}(cz+d)^{-2}\right]^2 = |cz+d|^{-4}.$$

From this we deduce

$$g^*(dxdy) = \Delta dxdy = |cz+d|^{-4}dxdy.$$

Furthermore,

$$g^*(y^{-2}) = \operatorname{Im}(\frac{az+b}{cz+d})^{-2} = y^{-2}|cz+d|^4$$

and therefore

$$g^*(d\mu_{\mathcal{H}}) = g^*(y^{-2}dxdy) = y^{-2}dxdy = d\mu_{\mathcal{H}}.$$

Now, the assertion follows from Lie Groups I, Theorem 1.9.

e) Show that $\mathcal{F} = \{z \in \mathcal{H} \mid (|z| > 1 \text{ and } -1/2 \le \operatorname{Re}(z) < 1/2) \text{ or } (|z| = 1 \text{ and } -1/2 \le \operatorname{Re}(z) \le 0)\}$ is a fundamental domain for the action of $G_{\mathbb{Z}}$ on $\mathcal{H}$. (Hint: For every $G_{\mathbb{Z}}$-orbit $G_{\mathbb{Z}}z$, $z \in \mathcal{H}$, consider $w \in G_{\mathbb{Z}}z$ with maximal imaginary part.)

Solution: First of all, note that

$$\operatorname{Im}(\frac{az+b}{cz+d}) = \frac{\operatorname{Im}(z)}{|cz+d|^2}.$$

If $c = 0$ then $\operatorname{Im}(z)/|cz+d|^2 = \operatorname{Im}(z)/|d|^2$ and $\operatorname{Im}(z)/|cz+d|^2 \le \operatorname{Im}(z)^{-1}|c|^2$ elsewise. Hence, given some $z \in \mathcal{H}$ the function

$$G_{\mathbb{Z}} \longrightarrow \mathcal{H},\ \gamma \mapsto \operatorname{Im}(\gamma z)$$

obtains a maximum on $G_{\mathbb{Z}}$, i.e. there exists $\gamma_0 \in G_{\mathbb{Z}}$ such that

$$\operatorname{Im}(\gamma_0 z) = \max_{\gamma \in G_{\mathbb{Z}}}\{\operatorname{Im}(\gamma z)\}.$$

Write $w = \gamma_0 z \in G_{\mathbb{Z}}z$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts as a translation $z \mapsto (z+1)$ on $\mathcal{H}$ we may assume that $-1/2 \le \operatorname{Re}(w) < 1/2$. In addition, since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts as inversion $z \mapsto -z^{-1}$ on $\mathcal{H}$

and

$$\mathrm{Im}(-\frac{1}{w}) = (-\frac{1}{w} + \frac{1}{\bar{w}})/(2i) = \frac{\mathrm{Im}(w)}{|w|^2}$$

one clearly has $|w| \geq 1$. It remains to show that we may impose $-1/2 \leq \mathrm{Re}(w) \leq 0$ if $|w| = 1$ but this follows easily from the fact that $z \mapsto -z^{-1}$ sends

$$\{z \in \mathcal{H} \mid (|z| = 1 \text{ and } 0 < \mathrm{Re}(z) \leq 1/2)\}$$

to

$$\{z \in \mathcal{H} \mid (|z| = 1 \text{ and } -1/2 \leq \mathrm{Re}(z) < 0)\}.$$

Thus, each orbit $G_{\mathbb{Z}}z$ intersects $\mathcal{F}$. To actually show that $\mathcal{F}$ is a fundamental domain it remains to prove that $\{z, \gamma z\} \subset \mathcal{F}$ for some $\gamma \in G_{\mathbb{Z}}$ implies $\gamma z = z$, i.e. $\gamma \in Stab(z)$. This can be done using considerations similar to those above and we leave these to the reader.

f) Show that the volume of $\mathcal{F}$ with respect to $\mu_{\mathcal{H}}$ is $\pi/3$. Deduce that $\mu(G/G_{\mathbb{Z}}) < \infty$.

Solution: This is a simple calculation:

$$\begin{aligned}\int_{\mathcal{F}} d\mu_{\mathcal{H}} &= \int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} y^{-2}dydx \\ &= \int_{-1/2}^{1/2} \left[-y^{-1}\right]_{y=\sqrt{1-x^2}}^{y=\infty} dx \\ &= \int_{-1/2}^{1/2} (1-x^2)^{-1/2}dx \\ &= [\arcsin(x)]_{x=-1/2}^{x=1/2} = \pi/3.\end{aligned}$$

The second assertion follows from Exercise 2 applied to $A = \mathcal{F}$.

g)* G/K looks like a (Riemannian globally) symmetric space. Give the geodesic symmetry s_i at i by using the formula we saw (but did not yet prove) in the lecture.

Solution: The geodesic symmetric s_i at i is given by $s_i(z) = -1/z$. The associated involution of G is $\sigma(g) = (g^{-1})^T$, where T means transposition of matrices (see Proposition 2.24 of the lecture notes).

h)* If you are courageous enough, deduce in 3f) that (normalizations[1] as in Exercise 4 below)

$$\mu(G/G_{\mathbb{Z}}) = \frac{\pi^2}{6} (= \zeta(2), \text{ where } \zeta(z) \text{ is the Riemann zeta function}).$$

A nice formula, isn't it? Hint: Be careful, the measure $\mu_{\mathcal{H}}$ is not what one gets "group-theoretically" by using the Haar measures μ_G and μ_K with their standard normalizations as below (, which you - as everyone else - should use).

Solution: Omitted.

Now, let us come back to the general case. First of all, we have to find a (bi-invariant) Haar measure on μ_G, which is not as easy as it is for $GL_N(\mathbb{R})$, where we can just write down a volume form. In the previous lecture last year, we gave a Haar measure for $SL_2(\mathbb{R})$ by

[1]Here, if G is a locally compact group and H a closed subgroup of G, then the (up to constant) unique non-zero semi-invariant measure on G/H is normalized by

$$\int_G f(g)\chi(g)^{-1}d\mu_G(g) = \int_{G/H}\left(\int_H f(gh)d\mu_H(h)\right)d\mu_{G/H}(gH).$$

decomposing it as a measure on the upper half plane $\mathcal{H}$ and on $K = SO_2(\mathbb{R})$. Indeed, the above exercise repeats parts of our proof there. In the general case, more refined tools are necessary, such as

4. **the Iwasawa decomposition of** $SL_n(\mathbb{R})$: In generalization of the groups in Exercise 3 above, we consider here the following Lie subgroups of $G = SL_n(\mathbb{R})$:

$$K = SO_n(\mathbb{R}), \text{ the special orthogonal group,}$$

$$P = \{(a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \in SL_n(\mathbb{R}) \mid a_{ij} = 0 \text{ if } i > j, a_{ii} > 0 \text{ for all } 1 \leq i \leq n\},$$

$$A = \{(a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \in SL_n(\mathbb{R}) \mid a_{ij} = 0 \text{ if } i \neq j,\ a_{ii} > 0 \text{ for all} 1 \leq i \leq n\}, \text{ and}$$

$$N = \{(n_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \in SL_n(\mathbb{R}) \mid n_{ij} = 0 \text{ if } i > j,\ n_{ii} = 1 \text{ for all } 1 \leq i \leq n\}.$$

a) Show (again) that

$$K \times P \longrightarrow G,\ (k, p) \longmapsto kp$$

and

$$A \times N \longrightarrow P,\ (a, n) \longmapsto an$$

are diffeomorphisms. (Hint: For a given $g \in G$, there exists a unique orthonormal basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ such that $ge_i = \alpha_{i1}v_1 + \ldots + \alpha_{ii}v_i$, $\alpha_{ii} > 0$, where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$.) Show that P is a semidirect product $A \ltimes N$.

Solution: The proof procedes as in Exercise 3 above by using the Gram-Schmidt algorithm.

b) Are the above diffeomorphisms group homomorphisms?

Solution: Counterexamples are legion.

5. **A construction of a Haar measure on** $SL_n(\mathbb{R})$

a) Show that $A \cong (\mathbb{R}^{>0})^{n-1}$ (as real Lie groups). Show that the volume form $x^{-1}dx$ on $(\mathbb{R}^{>0}, \cdot)$ gives a bi-invariant Haar measure. Write down the resulting bi-invariant Haar measure μ_A on A.

Solution: Any projection to $(n-1)$ diagonal entries gives an isomorphism $A \cong (\mathbb{R}^{>0})^{n-1}$. Now, $x^{-1}dx$ is a Haar measure on $(\mathbb{R}^{>0}, \cdot)$ because $l_\alpha^*(dx) = \alpha dx$, $l_\alpha^*(x^{-1}) = (\alpha x)^{-1}$ and hence

$$l_\alpha^*(x^{-1}dx) = x^{-1}dx.$$

Therefore, $x^{-1}dx$ is a bi-invariant Haar measure on the commutative group $(\mathbb{R}^{>0}, \cdot)$. One immediately infers from this that

$$a_{nn}da_{11}da_{22}\ldots da_{(n-1)(n-1)} = (a_{11}\ldots a_{(n-1)(n-1)})^{-1}da_{11}da_{22}\ldots da_{(n-1)(n-1)}$$

is a Haar measure on A.

b) There exists a canonical diffeomorphism $N \cong \mathbb{R}^{(n-1)(n-2)/2}$ by mapping a matrix the coefficients of its strictly upper triangle. Show that the Lebesque volume form $\prod_{i<j} dx_{ij}$ on $\mathbb{R}^{(n-1)(n-2)/2}$ gives a bi-invariant Haar measure on N. (Hint: From elementary linear algebra, you know that every matrix in N decomposes as a product of certain elementary matrices.)

Solution: Write $d\mu = \prod_{i<j} dx_{ij}$. It suffices to show that

$$l_g^*(d\mu) = r_g^*(d\mu) = d\mu,$$

where l_g (resp. r_g) is left- resp. right-multiplication in N by some $g \in N$. Since $l_{g'} \circ l_g = l_{g'g}$ and $r_{g'} \circ r_g = r_{gg'}$ it suffices to prove the above equality for a very elementary type of matrices, namely those $(n_{ij}) \in N$ satisfying

$$n_{ij} = \delta_{ij} + \alpha \delta_{(i+1)j} \text{ for arbitrary } \alpha \in \mathbb{R}.$$

We omit this calculation.

c)* Show that given locally compact groups A, N with bi-invariant Haar measures μ_A and μ_N a right Haar measure ν_P on $P = A \ltimes N$ is given by

$$d\nu_P(an) = \text{mod}_N(a) d\mu_A(a) d\mu_N(n),$$

where $\text{mod}_N(a)$ is the modulus of the automorphism $int(a) : n \mapsto ana^{-1}$ with respect to μ_N.

Solution: We have to prove that $r^*_{P,p'}(d\nu_P) = d\nu_P$ for any $p' = a'n' \in P$. For this note that

$$r_{P,p'}(an) = (r_{A,a'})(a)(r_{N,n'} \circ int(a')^{-1})(n).$$

Thus,

$$\begin{aligned} r^*_{P,p'}(d\nu_P(an)) &= (r_{A,a'})^*(\text{mod}_N(a) d\mu_A(a)) \cdot (r_{N,n'} \circ int(a')^{-1})^*(d\mu_N(n))) \\ &= \text{mod}_N(aa')\mu_A(a) \cdot \text{mod}_N(a')^{-1} d\mu_N(n) \\ &= \text{mod}_N(a)\mu_A(a) d\mu_N(n) \\ &= d\nu_P(an), \end{aligned}$$

which was to prove.

d) Let μ_K be a bi-invariant Haar measure on K, normalized such that $\mu_K(K) = 1$. Show that all non-zero positive Radon measures on G which are right-invariant for K and left-invariant for P must be (bi-invariant) Haar measures of G. Use this to show that

$$d\mu_G(kan) = d\mu_K(k)\mu_P(an) = \rho(a) d\mu_K(k) d\mu_A(a) d\mu_N(n),\ \rho(a) = \prod_{1 \le i < j \le n} \left(\frac{a_{ii}}{a_{jj}}\right)$$

gives a (bi-invariant) Haar measure on G.

Solution: This is just an application of the statement from c) and a computation of $\text{mod}_N(a)$ which we leave to the reader.

6. **Siegel sets**[2] **in** $SL_n(\mathbb{R})$: For every positive real numbers t, u we set

$$A_t = \{(a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in A \mid a_{ii} \le t a_{(i+1)(i+1)} \text{ for all } 1 \le i \le (n-1)\}, \text{ and}$$

$$N_u = \{(n_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in N \mid |n_{ij}| \le u\}.$$

Every set $\mathcal{S}_{t,u} = K A_t N_u$ is called a Siegel set.

a) $\mathcal{S}_{t,u}$ has finite measure for all $t, u > 0$.

Solution: $K A_t N_u$ is compact since A_t and N_u are compact.

b) Show that $N = N_{1/2} N_{\mathbb{Z}}$, where $N_{\mathbb{Z}} = \{(n_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in N \mid N_{ij} \in \mathbb{Z}\}$.

[2] named after Carl Ludwig Siegel (1896-1981), a number theorist, who pioneered the theory of automorphic forms of several variables. He also calculated $\mu(G/G_{\mathbb{Z}})$ explicitly for every natural n. In fact, he showed (with our Haar measure μ_G) that

$$\mu(G/G_{\mathbb{Z}}) = \zeta(2)\zeta(3)\ldots\zeta(n).$$

This can be done since $\mathcal{S}_{2/\sqrt{3},1/2}$ is not too far from being a fundamental domain for $G_{\mathbb{Z}}$. Finally, it is noteworthy that Siegel did not compute $\mu(G/G_{\mathbb{Z}})$ just for fun, as you might do nowadays, but in order to get information on the asymptotic growth of the number of certain lattice points.

Solution: We have to show that for any $(n_{ij}) \in N$ there exists $(z_{ij}) \in N_{\mathbb{Z}}$ satisfying

$$\left|\sum\nolimits_{1\leq k\leq n} n_{ik}z_{kj}\right| \leq \frac{1}{2}$$

for all i, j satisfying $1 \leq i < j \leq n$. Since

$$\sum\nolimits_{1\leq k\leq n} n_{ik}z_{kj} = \sum\nolimits_{i<k\leq n} n_{ik}z_{kj} + z_{ki}$$

one can arrange this by choosing the (z_{ij}) recursively for $i = (n-1), \ldots, 1$ (and all j simultaneously at each recursive step $i \mapsto (i-1)$).

c) Let $\|\cdot\|$ be the standard quadratic form on $\mathbb{R}^n$ and $e_1 = (1, 0, \ldots, 0)$. Argue that

$$\Phi : G \longrightarrow \mathbb{R}^\times,\ g \mapsto \|ge_1\|$$

is a continuous function on G. The function Φ attains a positive minimum on each $G_{\mathbb{Z}}$-orbit $gG_{\mathbb{Z}}$ in G.

Solution: It is only to show that Φ attains a minimum and this follows from $gG_{\mathbb{Z}}e_1 \subseteq g(\mathbb{Z}^n\backslash\{0\})$.

d) Show that this minimum must be attained at a point $gG_{\mathbb{Z}} \cap \mathcal{S}_{2/\sqrt{3},1/2}$. (Hint: $\Phi(kan) = a_{11}$, where $g = kan$ is the Iwasawa decomposition. It can attain these minima only for points $g \in G$ satisfying $a_{11} \leq (2/\sqrt{3})a_{22}$. Use this fact in an induction on n, the case $n = 1$ being clear (why?).)

Solution: We prove the assertion by induction on n. For this purpose we denote by $\mathcal{S}^{[i]}_{2/\sqrt{3},1/2}$ the respective Siegel set in $SL_i(\mathbb{R})$. We have

$$\mathcal{S}^{[1]}_{2/\sqrt{3},1/2} = G = \{(1)\}$$

and the assertion is obvious for $n = 1$. Now assume that the assertion is true for $(n-1)$; we prove it for n: Let $h \in gG_{\mathbb{Z}}$ be such that

$$\Phi(h) = \min_{\gamma\in G_{\mathbb{Z}}}\{\Phi(g\gamma)\} = \min_{\gamma\in G_{\mathbb{Z}}}\{\Phi(h\gamma)\}.$$

Write $h = kan$ for the Iwasawa decomposition of h. In addition,

$$k^{-1}h = \begin{pmatrix} a_{11} & * \\ 0 & b \end{pmatrix},\ b \in GL_{n-1}(\mathbb{R}),$$

and by the inductive assumption applied to $(a_{11})^{-1/n}b \in SL_{n-1}(\mathbb{R})$ there exists $\gamma_0^* \in SL_{n-1}(\mathbb{Z})$ such that

$$h^* = b\gamma_0^* \in \mathcal{S}^{[n-1]}_{2/\sqrt{3},1/2}.$$

Write $h^* = k^*a^*n^*$ for the Iwasawa decomposition of h^*. Then, setting $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^* \end{pmatrix}$ we obtain

$$k^{-1}h\gamma_0 = \begin{pmatrix} a_{11} & * \\ 0 & k^*a^*n^* \end{pmatrix},$$

and (by its uniqueness) the Iwasawa decomposition $k^{**}a^{**}n^{**}$ of $k^{-1}h\gamma_0$ is of the form

$$k^{**} = k\begin{pmatrix} 1 & 0 \\ 0 & k^* \end{pmatrix},\ a^{**} = k\begin{pmatrix} a_{11} & 0 \\ 0 & a^* \end{pmatrix} \text{ and } n^{**} = k\begin{pmatrix} n_{11} & 0 \\ 0 & n^* \end{pmatrix}.$$

By construction, $a_{ii} \leq (2/\sqrt{3})a_{i+1,i+1}$ for all $2 \leq i \leq n$ and $n^* \in N_{1/2}$. It mainly remains to prove that $a_{11} \leq (2/\sqrt{3})a_{22}$: Since γ_0 fixes e_1, we have $\Phi(h\gamma_0) = \Phi(h)$ and hence

$$\Phi(h\gamma_0) = \min_{\gamma \in G_{\mathbb{Z}}}\{\Phi(h\gamma_0\gamma)\}.$$

Write $h\gamma_0 = k_1 a_1 n_1$ for the Iwasawa decomposition and $n_1 = n_1' n_1''$, $n_1' \in N_{1/2}$, $n_1'' \in N_{\mathbb{Z}}$. One has

$$\Phi(h\gamma_0) = \Phi(k_1 a_1 n_1) = \Phi(a_1 n_1) = \Phi(a_1 n_1').$$

Let $\gamma \in G_{\mathbb{Z}}$ be the matrix

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 & 0 & \dots & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, $\Phi(a_1 n_1' \gamma) \leq \Phi(a_1 n_1')$ by assumption and hence

$$\Phi(h\gamma_0\gamma)^2 = [(a_1)_{22}]^2 + [(a_1)_{11}]^2 [(n_1)_{12}]^2 \leq [(a_1)_{11}]^2 = \Phi(h\gamma_0)^2.$$

From this we infer directly that $a_{11} \leq (2/\sqrt{3})a_{22}$.

e) Conclude that $G_{\mathbb{Z}}$ is a lattice.

Solution: This reduces to showing $\mu(\mathcal{S}_{2/\sqrt{3},1/2}) < \infty$. We give a more general proof showing that indeed $\mu(\mathcal{S}_{t,u}) < \infty$ for all $t, u > 0$.

$$\begin{aligned} \mu(\mathcal{S}_{t,u}) &= \int_{(K \times A_t \times N_u)} \rho(a)\, dk\, da\, dn \\ &= \mu_K(K)\mu_N(N_u) \int_{A_t} \rho(a)\, da. \end{aligned}$$

Since K is compact, $\mu_K(K) < \infty$; in fact $\mu_K(K) = 1$ by usual conventions. Equally, $\mu_N(N_u) < \infty$ since N_u is a compact subset of $\mathbb{R}^{(n-1)(n-2)/2}$. It remains to determine

$$\int_{A_t} \prod_{1 \leq i < j \leq n} \left(\frac{a_{ii}}{a_{jj}}\right) d\mu_A = \int_{A_t} \prod_{1 \leq i < j \leq n} \left(\frac{a_{ii}}{a_{jj}}\right) \frac{da_{11}}{a_{11}} \frac{da_{22}}{a_{22}} \cdots \frac{da_{(n-1)(n-1)}}{a_{(n-1)(n-1)}}.$$

There exist $\rho_i \in \mathbb{N}$, $i \in \{1, \dots, n\}$, such that

$$\prod_{1 \leq i < j \leq n} \left(\frac{a_{ii}}{a_{jj}}\right) = \prod_{1 \leq i < n} \left(\frac{a_{ii}}{a_{i+1,i+1}}\right)^{\rho_i}.$$

Now, the substitution $b_i = \log(a_{ii}/a_{i+1,i+1})$ gives

$$\int_{A_t} \prod_{1 \leq i < n} \left(\frac{a_{ii}}{a_{i+1,i+1}}\right)^{\rho_i} d\mu_A = \int_{-\infty}^{\log(t)} \cdots \int_{-\infty}^{\log(t)} \prod_{1 \leq i < n} \exp(\rho_i b_i)\, db_1 db_2 \dots db_{(n-1)} < \infty.$$