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Solution Exercise Sheet 2

Exercise 1.(Invariant Riemannian metrics on homogeneous spaces):

In the first exercise class we saw that every homogeneous G-manifold M is diffeomorphic to a quotient G/H, where $H = G_p < G$ is the stabilizer subgroup of a point $p \in M$. The diffeomorphism $F\colon G/H \to M$ is given by $F(gH) = g \cdot p$. Moreover, we saw that the set $R(M)^G$ of G-invariant Riemannian metrics on M can be identified with the set $\mathrm{Sym}_+(T_pM)^H$ of H-invariant inner products on the tangent space T_pM.

Complete our discussion by showing the following:

a) Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Then the differential $dF_e\colon \mathfrak{g}/\mathfrak{h} \cong T_eG/H \to T_pM$ induces a bijection between H-invariant inner products on T_pM and $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.

   Solution: Because $\mathrm{Ad}(H)$ preserves $\mathfrak{h}$, the adjoint action of H descends to an action on $\mathfrak{g}/\mathfrak{h}$. Moreover, $dF\colon \mathfrak{g}/\mathfrak{h} \cong T_{eH}G/H \to T_pM$ is H-equivariant. Indeed,

   $$\tilde{F}(hgh^{-1}) = hgh^{-1} \cdot p = hg \cdot p = h \cdot \tilde{F}(g)$$

   for all $g \in G, h \in H$, where $\tilde{F}(g) = g \cdot p$.
   Taking derivatives yields $d\tilde{F} \circ \mathrm{Ad}(h) = dh \circ d\tilde{F}$ which descends to $dF \circ \mathrm{Ad}(h) = dh \circ dF$ for all $h \in H$.
   It follows directly that $dF^*\colon \mathrm{Sym}_+(T_pM)^H \to \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(H)}$ is a bijection. □

b) Show that every $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle \in \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})$ is also $\mathrm{ad}(\mathfrak{h})$-invariant, i.e.

   $$\langle \mathrm{ad}(X)Y, Z\rangle + \langle Y, \mathrm{ad}(X)Z\rangle = 0$$

   for all $X \in \mathfrak{h}, Y, Z \in \mathfrak{g}/\mathfrak{h}$.
   If H is connected, the converse holds as well: Every $\mathrm{ad}(\mathfrak{h})$-invariant inner product is $\mathrm{Ad}(H)$-invariant.

   Solution: Let $\langle\cdot,\cdot\rangle \in \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})$ and let $X \in \mathfrak{h}, Y, Z \in \mathfrak{g}/\mathfrak{h}$.
   First, suppose $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(H)$-invariant. Then

   $$\langle \mathrm{Ad}(\exp(tX))Y, \mathrm{Ad}(\exp(tX))Z\rangle = \langle Y, Z\rangle,$$

and by differentiating at $t=0$ we obtain

$$\langle [X,Y],Z\rangle + \langle Y,[X,Z]\rangle = 0,$$

i.e. $\langle\cdot,\cdot\rangle \in \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})^{\mathrm{ad}(\mathfrak{h})}$.
Now suppose that H is connected and $\langle\cdot,\cdot\rangle$ is $\mathrm{ad}(\mathfrak{h})$-invariant, i.e.

$$\langle [X,Y],Z\rangle + \langle Y,[X,Z]\rangle = 0$$

holds for all $X\in\mathfrak{h}, Y,Z\in\mathfrak{g}/\mathfrak{h}$. In order to show that $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(H)$-invariant it will be sufficient to show that there is an open neighborhood $e\in U\subseteq H$ such that $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(U)$-invariant. Indeed, H is generated by U because H is connected.
Let $U\subseteq H$ be an open identity neighborhood onto which $\exp\colon V\subseteq\mathfrak{h}\to U\subseteq H$ is a diffeomorphism. Let $h\in U$ and $X\in V$ such that $h=\exp(X)$. We define $\psi(t):=\langle \mathrm{Ad}(\exp(tX))Y, \mathrm{Ad}(\exp(tX))Z\rangle$ for all $t\in\mathbb{R}$. Then $\psi(0)=\langle Y,Z\rangle$ and

$$\begin{aligned}
\psi'(t) &= \frac{d}{ds}\bigg|_{s=0} \langle \mathrm{Ad}(\exp((t+s)X))Y, \mathrm{Ad}(\exp((t+s)X))Z\rangle\\
&= \frac{d}{ds}\bigg|_{s=0} \langle \mathrm{Ad}(\exp(sX))\,\mathrm{Ad}(\exp(tX))Y, \mathrm{Ad}(\exp(sX))\,\mathrm{Ad}(\exp(tX))Z\rangle\\
&= \langle [X,\mathrm{Ad}(\exp(tX))Y], \mathrm{Ad}(\exp(tX))Z\rangle + \langle \mathrm{Ad}(\exp(tX))Y, [X,\mathrm{Ad}(\exp(tX))Z]\rangle\\
&= 0,
\end{aligned}$$

for all $t\in\mathbb{R}$.
Thus, $\psi(t)=\langle Y,Z\rangle$ for all $t\in\mathbb{R}$. In particular

$$\psi(1) = \langle \mathrm{Ad}(\exp(X))Y, \mathrm{Ad}(\exp(X))Z\rangle = \langle \mathrm{Ad}(h)Y,\mathrm{Ad}(h)Z\rangle = \langle Y,Z\rangle.$$

□

c) Let $G=\mathrm{GL}(n,\mathbb{R})$ and let $d_1,\dots,d_m\in\mathbb{N}$ such that $d_1+\dots+d_m=n$. Denote by $P<G$ the subgroup that consists of block upper triangular matrices of the form

$$\begin{pmatrix} \boxed{B_1} & & * \\ & \ddots & \\ 0 & & \boxed{B_m} \end{pmatrix},$$

where $B_i\in\mathrm{GL}(d_i,\mathbb{R})$, $i=1,\dots,m$.
Use the above characterization to show that there are no G-invariant Riemannian metrics on G/P.
<u>Remark:</u> The quotient space G/P can be interpreted as the flag variety of partial flags $\{0\}\subsetneq V_1\subsetneq V_2\subsetneq\dots\subsetneq V_m=\mathbb{R}^n$, where $\dim V_i = d_1+\dots+d_i$, $i=1,\dots,m$.

Solution: Let $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{R})=\mathrm{Lie}(G)$, let $\mathfrak{p}=\mathrm{Lie}(P)\le\mathfrak{g}$ and let $\mathfrak{l}\le\mathfrak{g}$ be the subspace of block lower diagonal matrices such that $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{l}$. Thus, we may

identify $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{l}$.
Furthermore, we use the standard notation $E_{ij} \in \mathfrak{g}$ to denote the matrix whose ij-entry is 1 and all other entries are 0.
Then $H := E_{11} - E_{nn} \in \mathfrak{p}$ and $E_{n1} \in \mathfrak{l} \cong \mathfrak{g}/\mathfrak{p}$. A direct computation yields

$$[H, E_{n1}] = [E_{11}, E_{n1}] - [E_{nn}, E_{n1}] = -E_{n1} - E_{n1} = -2E_{n1}.$$

If there were a G-invariant Riemannian metric on G/P then this would amount to an $\mathrm{ad}(\mathfrak{h})$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{l}$ by b). However, by the above computation, we would then have

$$0 = \langle [H, E_{n1}], E_{n1}\rangle + \langle E_{n1}, [H, E_{n1}]\rangle = -4\langle E_{n1}, E_{n1}\rangle \neq 0,$$

which is absurd. □

Exercise 2.(Compact Lie groups as symmetric spaces):

Let G be a compact connected Lie group and let

$$G^* = \{(g,g) \in G \times G : g \in G\} < G$$

denote the diagonal subgroup.

a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

   Solution: Consider the mapping $\sigma\colon (g_1, g_2) \mapsto (g_2, g_1)$. This is an involutive automorphism of the product group $G \times G$. The fixed set of σ is precisely the diagonal G^*. It follows that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair. The coset space $G \times G/G^*$ is diffeomorphic to G under the mapping $[(g_1, g_2)] \mapsto \pi(g_1, g_2) := g_1 g_2^{-1}$. □

b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

   Solution: By Proposition 3.4 from Helgason, Ch. IV, we see that G is a Riemanian globally symmetric space in each bi-invariant Riemannian structure; note here that a Riemannian structure on $G \times G/G^*$ is $G \times G$-invariant if and only if the corresponding Riemannian structure on G is bi-translation invariant. □

c) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the exponential map from $\mathfrak{g}$ into the Riemannian *globally symmetric space G*.

   Solution: Note that the product algebra $\mathfrak{g} \times \mathfrak{g}$ is the Lie algebra of $G \times G$.

Let $\exp^*$ denote the exponential mapping of $\mathfrak{g} \times \mathfrak{g}$ int $G \times G$, exp denote the exponential mapping of $\mathfrak{g}$ into G, and Exp denote the Riemannian exponential mapping of $\mathfrak{g} \cong T_e G$ into G (considered as a Riemannian globally symmetric space). We want to show that $\exp X = \operatorname{Exp} X$ for all $X \in \mathfrak{g}$. Using $d\pi(X,Y) = X - Y$, we deduce that $\pi(\exp^*(X,-X)) = \operatorname{Exp}(d\pi(X,-X))$. Hence $\exp X \cdot (\exp(-X))^{-1} = \operatorname{Exp}(2X)$ and this implies that $\exp X = \operatorname{Exp} X$. □

Exercise 3.(Compact semisimple Lie groups as symmetric spaces):

A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the Killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

Solution: Let π and σ be as in the above solution. The map $d\pi$ maps the -1 eigenspace of $d\sigma$ onto $\mathfrak{g}$ as follows: $d\pi(X,-X) = 2X$. Using this, we can check that

$$2B_{\mathfrak{g}\times\mathfrak{g}}((X,-X),(X,-X)) = B_{\mathfrak{g}}(2X,2X),$$

which is equivalent to $Q = 2Q^*$. □

Exercise 4.(Constant sectional curvature determines isometry type):

Show or look up the following theorem:

Let M be a complete and simply connected Riemannian manifold of dimension n and constant sectional curvature K. Then M is isometric to:

a) the hyperbolic n-space $\mathbb{H}^n$, if $K \equiv -1$;

b) the Euclidean n-space $\mathbb{R}^n$, if $K \equiv 0$;

c) the n-sphere $\mathbb{S}^n$, if $K \equiv 1$.

Solution: This is Theorem 4.1 from do Carmo, Ch. 8. □

Exercise 5.(Closed differential forms):

Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\mathrm{Isom}(M)^\circ$. Prove that $d\omega = 0$.

Solution: Let s_m denote the geodesic symmetry at some point $m \in M$, and let $\omega \in \Omega^p(M)$ be an invariant differential p-form on M. Because $d_m s_m = -\mathrm{id}\colon T_pM \to T_pM$, we get $(s_m^*\omega)_m = (-1)^p \omega_m$ at the point $m \in M$. Because ω is invariant, $s_m^*\omega$ is invariant as well. Because $\mathrm{Iso}(M)^\circ$ acts transitively, invariant differential forms are determined by their value at a single point such that

$$s_m^*\omega = (-1)^p \omega$$

on all of M.

Therefore, we obtain

$$d\omega = (-1)^p d(s_m^*\omega) = (-1)^p s_m^* d\omega = (-1)^{2p+1} d\omega,$$

whence $d\omega = 0$. $\square$