Solution Exercise Sheet 4

Exercise 1.(Complexification and Killing form):

Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let K_0, K and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}, \mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

a) $K_0(X,Y) = K(X,Y)$ for all $X, Y \in \mathfrak{l}_0$;

b) $K^{\mathbb{R}}(X,Y) = 2 \cdot \mathrm{Re}(K(X,Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.

Solution: The first relation is obvious. For the second let $\mathcal{B} := \{X_i : i = 1, \ldots, n\}$ be a basis of $\mathfrak{l}$. Let $X, Y \in \mathfrak{l}$. Then we may write

$$\mathrm{ad}(X)\mathrm{ad}(Y)(X_i) = \sum_{j=1}^{n} \alpha_{ij} \cdot X_j, \qquad i = 1, \ldots, n, \tag{1}$$

for some complex numbers $\alpha_{ij} = \beta_{ij} + i \cdot \gamma_{ij} \in \mathbb{C}$. Denote by A, B, C the $n \times n$-matrices with entries α_{ij}, β_{ij}, γ_{ij}, respectively. Then A is the matrix representation of $\mathrm{ad}(X)\mathrm{ad}(Y)$ with respect to the basis $\mathcal{B}$

$$M_{\mathcal{B}}(\mathrm{ad}(X)\mathrm{ad}(Y)) = A = B + iC$$

and B, C are the real, imaginary parts of A. Now, consider the basis $\mathcal{C} = \{X_1, \ldots, X_n, iX_1, \ldots, iX_n\}$ of $\mathfrak{l}^{\mathbb{R}}$. Then

$$\mathrm{ad}(X)\mathrm{ad}(Y)(iX_i) = \sum_{j=1}^{n} -\gamma_{ij} \cdot X_j + \sum_{j=1}^{n} \beta_{ij} \cdot (iX_j), \qquad i = 1, \ldots, n, \tag{2}$$

and with (1) we obtain that the matrix representation of $\mathrm{ad}(X)\mathrm{ad}(Y)$ with respect to the basis $\mathcal{C}$ is given by

$$A' := M_{\mathcal{C}}(\mathrm{ad}(X)\mathrm{ad}(Y)) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Thus

$$2\,\mathrm{Re}\,K(X,Y) = 2\,\mathrm{Re}(\mathrm{tr}\,A) = 2B = \mathrm{tr}\,A' = K^{\mathbb{R}}(X,Y).$$

$\square$

Exercise 2.(Semisimple OSLAs):

Let $(\mathfrak{l},\Theta)$ be an orthogonal symmetric Lie algebra with $\mathfrak{l}$ semisimple. Show that:

a) $\mathfrak{u}$ equals its normalizer in $\mathfrak{l}$;

b) if $\mathfrak{u}$ contains no ideal in $\mathfrak{l}$ then $[\mathfrak{e},\mathfrak{e}] = \mathfrak{u}$.

Solution: To (a): Decompose $\mathfrak{l} = \mathfrak{u} + \mathfrak{e}$. It will suffice to show that if $X \in \mathfrak{e}$ is in the normalizer of $\mathfrak{u}$ then $X = 0$.

To that end note that $[X,\mathfrak{u}] \subseteq [\mathfrak{e},\mathfrak{u}] \subseteq \mathfrak{e}$ and also $[X,\mathfrak{u}] \subseteq \mathfrak{u}$ since X is in the normalizer of $\mathfrak{u}$. Thus $[X,\mathfrak{u}] \in \mathfrak{u} \cap \mathfrak{e} = 0$, i.e. X commutes with every element of $\mathfrak{u}$. Let us further decompose $X = X_0 + X_- + X_+ \in \mathfrak{e}_0 + \mathfrak{e}_- + \mathfrak{e}_+$. Since $\mathfrak{e}_0$, $\mathfrak{e}_-$ and $\mathfrak{e}_+$ are invariant under $\operatorname{ad}(\mathfrak{u})$ we have

$$0 = [X,\mathfrak{u}] = \underbrace{\operatorname{ad}(\mathfrak{u})(X_0)}_{\subseteq \mathfrak{e}_0} + \underbrace{\operatorname{ad}(\mathfrak{u})(X_+)}_{\subseteq \mathfrak{e}_+} + \underbrace{\operatorname{ad}(\mathfrak{u})(X_-)}_{\subseteq \mathfrak{e}_-},$$

such that $[X_0,\mathfrak{u}] = [X_-,\mathfrak{u}] = [X_+,\mathfrak{u}] = 0$. It follows that

$$\operatorname{ad}(X_+)\operatorname{ad}(X_+)\mathfrak{u} = 0 \quad \text{and} \quad \operatorname{ad}(X_+)\operatorname{ad}(X_+)\mathfrak{e} = [X_+, \underbrace{[X_+,\mathfrak{e}]}_{\subseteq \mathfrak{u}}] = 0,$$

whence $\operatorname{ad}(X_+)^2 \equiv 0$, and $\operatorname{ad}(X_-)^2 \equiv 0$, analogously. Thus

$$B(X_+,X_+) = \operatorname{tr}\operatorname{ad}(X_+)^2 = 0 = B(X_-,X_-)$$

and $X_+ = X_- = 0$, since $B|_{\mathfrak{e}_+\times\mathfrak{e}_+} \gg 0$ and $B|_{\mathfrak{e}_-\times\mathfrak{e}_-} \ll 0$. Therefore $X = X_0 \in \mathfrak{e}_0$. However, $\mathfrak{l}$ is semisimple and $\mathfrak{e}_0$ is an abelian ideal whence $\mathfrak{e}_0 = 0$.

Remark: The above argument also shows that we have in every orthogonal symmetric Lie algebra $\mathfrak{l}$ (not necessarily semisimple):

If $X \in \mathfrak{e}$ commutes with every element in $\mathfrak{u}$ then $X \in \mathfrak{e}_0$ (see [**?**, Chapter V, Corollary 1.7]).

To (b): We decompose $\mathfrak{e} = \mathfrak{e}_0 + \mathfrak{e}_- + \mathfrak{e}_+$ and $\mathfrak{u} = \mathfrak{u}_0 + \mathfrak{u}_- + \mathfrak{u}_+$. By our assumption $\mathfrak{e}_0 = 0$ such that $[\mathfrak{u}_0,\mathfrak{e}] = [\mathfrak{u}_0,\mathfrak{e}_+] + [\mathfrak{u}_0,\mathfrak{e}_-] = 0$ whence $\mathfrak{u}_0$ is an ideal in $\mathfrak{l}$ so that $\mathfrak{u}_0 = 0$ by our hypothesis. Finally,

$$[\mathfrak{e},\mathfrak{e}] = [\mathfrak{e}_+,\mathfrak{e}_+] + [\mathfrak{e}_-,\mathfrak{e}_-] = \mathfrak{u}_+ + \mathfrak{u}_- = \mathfrak{u}$$

because $[\mathfrak{e}_+,\mathfrak{e}_-] = 0$. □

Exercise 3.($\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$):

Exhibit an explicit isomorphism between $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$.

Hint: Consider the the vector space V of 2×2-skew-Hermitian matrices and endow it with the quadratic form $q(v) := \det(v)$. Now, let $\mathrm{SL}(2,\mathbb{C})$ act on V via $g.v := gv\bar{g}^t$.

Solution: Every element $v \in V = \{v \in \mathbb{C}^{2\times 2} : \bar{v}^t = -v\}$ can be written as

$$v = \begin{pmatrix} i(x_1 - x_3) & -x_2 + ix_4 \\ x_2 + ix_4 & i(x_1 + x_3) \end{pmatrix}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. We compute

$$q(v) = \det(v) = -x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

It is readily verified that the given action of $\mathrm{SL}(2,\mathbb{C})$ on V is well-defined and preserves q. Indeed,

$$q(g.v) = \det(gv\bar{g}^t) = \det(g)\det(v)\det(\bar{g})^t = \det(v)$$

for every $g \in \mathrm{SL}(2,\mathbb{C})$ and every $v \in V$.

Thus we obtain a Lie group homomorphism $\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)^\circ$, $\phi(g)(v) = g.v$. It is easy to check that $\{\pm I\} \subseteq \ker\phi$. Further, if $\phi(g) = I$ then in particular

$$g\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\bar{g}^t = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$
$$g\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\bar{g}^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$g\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\bar{g}^t = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and it is elementary to deduce that $g = \pm I$. Hence, $\ker\phi = \{\pm I\}$ and in particular ϕ is injective on a neighbourhood of I. Because it is a Lie group homomorphism and therefore has constant rank, its differential $d\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ is injective. Both Lie algebras have real dimension 6 such that $d\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ gives indeed the sought for isomorphism. □

Exercise 4.(Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$):

Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

Solution: Recall that we have seen in the lecture that $(\mathrm{SO}(n+1), \mathrm{SO}(n), \sigma)$ and $(\mathrm{SO}(1,n)^\circ, \mathrm{SO}(n), \sigma)$ are Riemannian symmetric pairs where $\sigma(g) := I_{1,n} g I_{1,n}$ in both cases. Further we have seen that the associated symmetric spaces $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are isometric to the n-sphere $\mathbb{S}^n$ and (real) hyperbolic n-space $\mathbb{H}^n$. These have $(\mathfrak{so}(n+1), \zeta)$ and $(\mathfrak{so}(1,n), \zeta)$ as orthogonal symmetric Lie algebras, respectively, where $\zeta(X) = d\sigma(X) = I_{1,n} X I_{1,n}$ in both cases.

We have also seen in the lecture that the orthogonal symmetric Lie algebras $(\mathfrak{so}(p+q), \zeta_{p,q})$ and $(\mathfrak{so}(p,q), \zeta_{p,q})$ are dual to each other for all $p, q \geq 1$ where $\zeta_{p,q}(X) = I_{p,q} X I_{p,q}$ in both cases. Thus for $p = 1, q = n$ we obtain the assertion. □