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Symmetric Spaces

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Solution Exercise Sheet 5

Exercise 1.(Maximal abelian subspaces and regular elements in $\mathfrak{sl}(n,\mathbb{R})$):

Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. A Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X = X^t\}$ and $\mathfrak{k} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X = -X^t\}$. Define

$$\mathfrak{a} = \left\{ \mathrm{diag}(t_1,\dots,t_n) : t_j \in \mathbb{R}, \sum_{j=1}^{n} t_j = 0 \right\}.$$

a) Prove that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.

Solution: It is easy to verify that $\mathfrak{a}$ is indeed an abelian subspace of $\mathfrak{p}$. Therefore, it remains to show that $\mathfrak{a}$ is indeed maximal. Let $\mathfrak{a}' \supseteq \mathfrak{a}$ be an abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$. Let $Y \in \mathfrak{a}'$. Then

$$0 = [Y,X] = YX - XY$$

for every $X = \mathrm{diag}(t_1,\dots,t_n) \in \mathfrak{a}$, or equivalently

$$y_{ij}t_i = y_{ij}t_j \qquad \forall i,j = 1,\dots,n \tag{1}$$

for every $X = \mathrm{diag}(t_1,\dots,t_n) \in \mathfrak{a}$. Choosing $t_1,\dots,t_n$ to be pairwise distinct then implies that $y_{ij} = 0$ for all $i \neq j$ so that $Y \in \mathfrak{a}$. That shows that $\mathfrak{a}' = \mathfrak{a}$ and $\mathfrak{a}$ is maximal. □

b) Prove (without appealing to the general theorem) any maximal abelian subspace of $\mathfrak{p}$ is of the form $S\mathfrak{a}S^{-1}$ where $S \in SO(n)$.

Solution: Let $\mathfrak{a}'$ be a maximal abelian subspace of $\mathfrak{p}$ with basis $\{Y_1,\dots,Y_r\}$. All of the Y_i commute pairwise, whence there is an element $S \in O(n)$ that diagonalizes all of them simultaneously, i.e. $SY_iS^{-1} = D_i$ for every $i = 1,\dots,r$ where D_i is some traceless diagonal matrix. Because we are free to multiply S with $\mathrm{diag}(-1,1,\dots,1)$ we may assume that $S \in \mathrm{SO}(n)$. It follows, that $S\mathfrak{a}'S^{-1} \subseteq \mathfrak{a}$ and due to maximality $S\mathfrak{a}'S^{-1} = \mathfrak{a}$. □

c) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct.

Solution: Let $X \in \mathfrak{p}$ be a regular element, i.e. $C_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian. By part (b) there is $S \in \mathrm{SO}(n)$ such that

$$\mathfrak{a} = S(C_{\mathfrak{g}}(X) \cap \mathfrak{p})S^{-1} = C_{\mathfrak{g}}(SXS^{-1}) \cap \mathfrak{p}. \tag{2}$$

We have used here that $K = \mathrm{SO}(n)$ acts via the adjoint representation $\mathrm{Ad}(S)X = SXS^{-1}$ which is by Lie algebra automorphisms preserving the Cartan decomposition. Then $SXS^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) =: D$ is a diagonal matrix and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X. Let $P_{ij} \in \mathfrak{p}$ denote the $n \times n$-permutation-matrix that permutes the canonical basis vectors $e_i \leftrightarrow e_j$ for all $i \neq j$ and fixes the rest. Then

$$[D, P_{ij}](e_k) = DP_{ij}e_k - P_{ij}De_k = 0$$

for every $k \neq i, j$,

$$[D, P_{ij}](e_i) = DP_{ij}e_i - P_{ij}De_i = (\lambda_j - \lambda_i)e_j$$

and

$$[D, P_{ij}](e_j) = DP_{ij}e_j - P_{ij}De_i = (\lambda_i - \lambda_j)e_j.$$

Thus $P_{ij} \in C_{\mathfrak{g}}(D) \cap \mathfrak{p}$ if $\lambda_i = \lambda_j$. However, $P_{ij} \notin \mathfrak{a}$ which contradicts (2).

Conversely, let $X \in \mathfrak{p}$ have distinct eigenvalues. By a theorem of the lecture we know that X is contained in a maximal abelian subspace $\mathfrak{a}'$ and there is $S \in \mathrm{SO}(n)$ such that $S\mathfrak{a}'S^{-1} = \mathfrak{a}$. Note that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = SXS^{-1}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X. Obviously, $C_{\mathfrak{g}}(D) \cap \mathfrak{p} \supseteq \mathfrak{a}$. Now, let $Y \in C_{\mathfrak{g}}(D) \cap \mathfrak{p}$. Then by the same computation leading towards (1), we obtain $y_{ij}\lambda_i = y_{ij}\lambda_j$ for all $i, j = 1, \ldots, n$. Since the eigenvalues of X are distinct that implies that $y_{ij} = 0$ for $i \neq j$, whence $C_{\mathfrak{g}}(D) \cap \mathfrak{p} \subseteq \mathfrak{a}$. □

Exercise 2.(Maximal abelian subspaces and regular elements in $\mathfrak{sp}(2n, \mathbb{R})$):

Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$. Recall that a Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = A^t, B = B^t \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A = -A^t, B = B^t \right\}.$$

a) Define

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \mathrm{diag}(t_1, \ldots, t_n). \right\}.$$

Prove that A is a maximal abelian subspace of $\mathfrak{p}$.

Solution: As in exercise 1 it is immediate to check that $\mathfrak{a}$ is abelian. It remains to show that it is maximal abelian. Let $\mathfrak{a}' \supseteq \mathfrak{a}$ be an abelian subspace of $\mathfrak{p}$

containing $\mathfrak{a}$. Let $Y = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{a}'$. Then

$$0 = [Y,X] = \begin{pmatrix} AD - DA & -BD - DB \\ BD + DB & AD - DA \end{pmatrix} = \begin{pmatrix} [A,D] & -BD - DB \\ BD + DB & [A,D] \end{pmatrix}$$

for every $X = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in \mathfrak{a}$. As in 1a) it follows that A is diagonal. Further, $BD + DB = 0$ is equivalent to

$$b_{ij}(\lambda_i + \lambda_j) = 0 \qquad \forall i,j = 1,\dots,n$$

where $D = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$ so that $B = 0$ for an appropriate choice of $\lambda_1,\dots,\lambda_n$. That implies that $Y \in \mathfrak{a}$, whence $\mathfrak{a}' = \mathfrak{a}$, and $\mathfrak{a}$ is indeed maximal. □

b) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct and non-zero.

Solution: The proof is essentially the same as for $\mathfrak{sl}(n,\mathbb{R})$. Suppose $X \in \mathfrak{p}$ is regular, i.e. $C_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian. Then there is $k \in K := \mathrm{SO}(2n,\mathbb{R}) \cap \mathrm{Sp}(2n,\mathbb{R})$ such that

$$kC_{\mathfrak{g}}(X) \cap \mathfrak{p}k^{-1} = C_{\mathfrak{g}}(kXk^{-1}) \cap \mathfrak{p} = \mathfrak{a}.$$

Write

$$kXk^{-1} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

with $D = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$. Note that $\lambda_1,\dots,\lambda_n,-\lambda_1,\dots,-\lambda_n$ are the eigenvalues of X. It is easy to verify, that if $\lambda_i = \lambda_j$ for $i \neq j$ then

$$\begin{pmatrix} P_{ij} & 0 \\ 0 & -P_{ij} \end{pmatrix} \in C_{\mathfrak{g}}(X) \cap \mathfrak{p}$$

but not in $\mathfrak{a}$, and if $\lambda_i = 0$ then

$$\begin{pmatrix} 0 & E_{ii} \\ E_{ii} & 0 \end{pmatrix} \in C_{\mathfrak{g}}(X) \cap \mathfrak{p}$$

but not in $\mathfrak{a}$. Both contradicts X being regular such that $\lambda_1,\dots,\lambda_n$ are all pairwise distinct and non-zero.

Conversely, let $X \in \mathfrak{p}$ have distinct and non-zero eigenvalues. By a theorem of the lecture we know that X is contained in a maximal abelian subspace $\mathfrak{a}'$ and there is $k \in K$ such that $k\mathfrak{a}'k^{-1} = \mathfrak{a}$. Note that

$$kXk^{-1} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} =: T$$

where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$, are the eigenvalues of X. Obviously, $C_{\mathfrak{g}}(T) \cap \mathfrak{p} \supseteq \mathfrak{a}$. Now, let $Y \in C_{\mathfrak{g}}(T) \cap \mathfrak{p}$. Then by the same computation as in (a), we obtain $y_{ij} = 0$ for all $i \neq j$, whence $C_{\mathfrak{g}}(T) \cap \mathfrak{p} \subseteq \mathfrak{a}$. □

Exercise 3.(The Siegel upper half space):

Let

$$\mathbb{H}_n := \{Z \in \mathbb{C}^{n\times n} : Z = Z^t,\ \operatorname{Im}(Z) \text{ is positive-definite}\}.$$

Find an explicit isomorphism between $\mathbb{H}_n$ and $\operatorname{Sp}(2n, \mathbb{R})/(\operatorname{SO}(2n) \cap \operatorname{Sp}(2n, \mathbb{R}))$. Use this and the previous exercise to construct a maximal flat of $\mathbb{H}_n$.

Hint: Consider the map

$$\phi : \operatorname{Sp}(2n, \mathbb{R}) \to \mathbb{H}_n,$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (Ai + B) \cdot (Ci + D)^{-1}.$$

Solution: The space $\mathbb{H}_n$ is also called the *Siegel upper half space*. Before we solve the problem we will show that the symplectic group $G := \operatorname{Sp}(2n, \mathbb{R})$ acts transitively via *generalized Möbius transformations*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \star Z := (AZ + B)(CZ + D)^{-1}$$

on $\mathbb{H}_n$ and that $i \cdot I_n$ has stabilizer $K := \operatorname{Sp}(2n, \mathbb{R}) \cap \operatorname{SO}(2n, \mathbb{R})$.

In order to do so, let us first see that our definition makes sense, i.e. that $(CZ + D)$ is indeed invertible and $g \star Z \in \mathbb{H}_n$ for every $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}(2n, \mathbb{R}), Z \in \mathbb{H}_n$. Put

$$P := AZ + B, \quad Q := CZ + D.$$

Recall that $g \in \operatorname{Sp}(2n, \mathbb{R})$ is equivalent to $A^tC = C^tA$, $B^tD = D^tB$ and $A^tD - C^tB = I$. Then

$$\begin{aligned} P^t\bar{Q} - Q^t\bar{P} &= (ZA^t + B^t)(C\bar{Z} + D) - (ZC^t + D^t)(A\bar{Z} + B) && (3)\\ &= ZA^tC\bar{Z} + ZA^tD + B^tC\bar{Z} + B^tD && (4)\\ &\quad - (ZC^tA\bar{Z} + ZC^tB + D^tA\bar{Z} + D^tB) && (5)\\ &= Z - \bar{Z} = 2i \operatorname{Im} Z. && (6) \end{aligned}$$

Suppose $\xi \in \mathbb{C}^n$ is in the kernel of Q. Then

$$\xi^t \operatorname{Im} Z \bar{\xi} = \frac{1}{2i}\left(\xi^t P^t \bar{Q}\bar{\xi} - \xi^t Q^t \bar{P}\xi\right) = 0.$$

Since $\operatorname{Im} Z$ is positive definite, that implies that $\xi = 0$. Therefore, Q is invertible.

That $g \star Z$ is symmetric is equivalent to

$$P^t Q = (Z^t A^t + B^t)(CZ + D) = (Z^t C^t + D^t)(AZ + B) = Q^t P$$

which follows again from g being symplectic.

Also

$$\begin{aligned}\operatorname{Im}(g \star Z) &= \frac{1}{2i}\left(PQ^{-1} - \bar{P}\bar{Q}^{-1}\right)\\ &= \frac{1}{2i}\left(\left(Q^{-1}\right)^t P^t - \bar{P}\bar{Q}^{-1}\right)\end{aligned}$$

is positive definite if and only if

$$\begin{aligned}Q^t \operatorname{Im}(g \star Z)\bar{Q} &= \frac{1}{2i}\left(Q^t\left(Q^{-1}\right)^t P^t \bar{Q} - Q^t \bar{P}\bar{Q}^{-1}\bar{Q}\right)\\ &= \frac{1}{2i}\left(P^t \bar{Q} - Q^t \bar{P}\right) = \operatorname{Im} Z\end{aligned}$$

is positive definite by (6).

Let us now verify that $\star$ is an action. It is immediate that $I \star Z = Z$. Let

$$g = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad h = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}).$$

Then

$$\begin{aligned}g \star (h \star Z) &= ((A_1 A_2 Z + A_1 B_2)(C_2 Z + D_2)^{-1} + B_1)\\ &\quad \cdot ((C_1 A_2 Z + C_1 B_2)(C_2 Z + D_2)^{-1} + D_1)^{-1}\\ &= ((A_1 A_2 Z + A_1 B_2) + B_1(C_2 Z + D_2))\cancel{(C_2 Z + D_2)^{-1}}\\ &\cdot \cancel{(C_2 Z + D_2)}((C_1 A_2 Z + C_1 B_2) + D_1 C_2 Z + D_1 D_2)^{-1}\\ &= (g \cdot h) \star Z.\end{aligned}$$

That finishes the proof that the symplectic group acts via generalized Möbius transformations on the Siegel upper half space.

Now, let us see that the action is transitive. Indeed, let $Z = X + iY \in \mathbb{H}_n$. Then the matrices

$$g = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix}, \quad h = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

are symplectic and it is readily verified that $(hg)\star iI = X + iY$.

Finally, suppose that
$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$$
stabilizes iI. Then $iI = (Ai+B)(Ci+D)^{-1}$ which is equivalent to $B = -C$ and $A = D$. Because g is symplectic we obtain $I = A^tA + B^tB$ and $A^tB = B^tA$. Therefore
$$g^tg = \begin{pmatrix} A^t & -B^t \\ B^t & A^t \end{pmatrix}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A^tA + B^tB & A^tB - B^tA \\ B^tA - A^tB & B^tB + A^tA \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$
whence $g \in \mathrm{SO}(2n,\mathbb{R})$, and $\mathrm{stab}(iI) \subseteq K$. Vice versa, let
$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K.$$

From $g^tg = I$ we obtain
$$A^tA + C^tC = I = B^tB + D^tD, \qquad 0 = A^tB + C^tD(= B^tA + D^tC)$$
and because g is symplectic
$$A^tC = C^tA, \qquad B^tD = D^tB, \qquad A^tD - C^tB = I.$$

In particular,
$$I = A^tD - C^tB + iA^tB + iC^tD = A^t(D+iB) + C^t(Di - B) \tag{7}$$
and
$$I = D^tD + B^tB = (D^t + iB^t)(D - iB),$$
whence $(D+iB)^{-1} = D^t - iB^t$. Using that relation in (7) we obtain
$$D^t - iB^t = A^t + C^t(iD - B)(D^t - iB^t) = A^t + iC^t$$
so that $A = D$ and $B = -C$. That is equivalent to $g\star iI = iI$.

This shows that $\mathbb{H}_n$ is diffeomorphic to $\mathrm{Sp}(2n,\mathbb{R})/(\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{SO}(2n,\mathbb{R}))$ via the map
$$\phi : G/K \to \mathbb{H}_n, gK \mapsto g\star iI.$$

By exercise 2) a maximal abelian subspace of $\mathfrak{p}$ is given by
$$\mathfrak{a} = \left\{\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \mathrm{diag}(t_1,\dots,t_n).\right\}.$$

Thus a maximal flat subspace of G/K is given by $\exp(\mathfrak{a})K$. Note that

$$\exp(\mathfrak{a}) = \left\{ \begin{pmatrix} \operatorname{diag}(\lambda_1,\ldots,\lambda_n) & 0 \\ 0 & \operatorname{diag}(\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_n}) \end{pmatrix} : \lambda_1,\ldots,\lambda_n > 0 \right\}.$$

Therefore,

$$\begin{aligned} \phi(\exp(\mathfrak{a})K) &= \{g \star iI : g \in \exp\mathfrak{a}\} \\ &= \{i \operatorname{diag}(\lambda_1^2,\ldots,\lambda_n^2) : \lambda_i > 0\} \end{aligned}$$

is a maximal flat subspace of $\mathbb{H}_n$ via the identification $\phi : G/K \to \mathbb{H}_n$. □

Exercise 4.(Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$):

Let $V = \mathbb{C}[X,Y]$ be the vector space of polynomials in two variables. Let V_m denote the vector subspace of all homogeneous polynomials of degree m. This has a basis given by the monomials $X^m, X^{m-1}Y, \ldots, Y^m$. We turn this vector subspace into a module for $\mathfrak{sl}(2,\mathbb{C})$ by defining a Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_m)$ in the following way

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X\frac{\partial}{\partial Y}, \quad \phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = Y\frac{\partial}{\partial X}, \quad \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Show that this defines an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

Solution: Put

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$E' = \phi(E) = X\frac{\partial}{\partial Y}, \quad F' = \phi(F) = Y\frac{\partial}{\partial X}, \quad H' = \phi(H) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

One easily checks that

$$[E,F] = H, \quad [E,H] = -2E, \quad [F,H] = 2F,$$

and

$$[E',F'] = H', \quad [E',H'] = -2E', \quad [F',H'] = 2F',$$

so that ϕ defines a Lie algebra homomorphism.

It remains to be shown that ϕ is irreducible. Suppose that there is a non-trivial invariant subspace $0 \lneq V' \lneq V_m$, and let $v' \in V'$ be non-zero. Since $\deg_Y(E'v) <$

$\deg_Y v$ for all $v \in V_m$, there is a minimal $k \in \mathbb{N}$ such that $E'^{k+1} v' = 0$. Then $0 \neq E'^k v' \in \ker E' = \mathbb{C} X^m$, i.e. $E'^k v' = \alpha X^m$ for some non-zero $\alpha \in \mathbb{C}$. Now, applying F' successively to αX^m yields the full basis $\{X^m, X^{m-1}Y, \dots, XY^{m-1}, Y^m\}$ which is contained in V' by invariance. However, that implies that $V' = V_m$ contradicting our assumption. $\square$