Lecture

24 Feb 2021

$\boxed{\text{Symmetric spaces}}$

(1) "historical introduction" of symm. spaces and examples
(2) Riemannian characterization
(3) Lie theoretical characterization
(4) Relation between the two
(5) Classification of symm. space w/

"Differential geometry, Lie gps and symmetric spaces", S. Helgason
"Geometry of neg. curved spaces", P. Eberlein

I. "Historical intro"

Élie Cartan 1926 : clasif. of "spaces s.t. central symmetries preserve distances" = clasif. of cplx forms of s.s. Lie algebras.

Original defn: the curvature tensor has vanishing covariant derivative.

Three reasons to study symm. sp.
(1) synergy between Lie th. & Riem. geom.
(2) Many known ex. are symm. sp.
(3) beautiful!

Examples

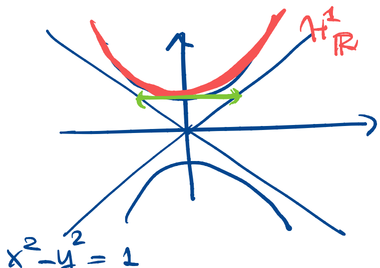
(1) Euclidean space $\mathbb{E}^n := (\mathbb{R}^n, g_{eucl})$ is a symm. sp. of vanishing sect. curvature. $Iso(\mathbb{E}^n) = O(n,\mathbb{R}) \ltimes \mathbb{R}^n$

(2) $\mathbb{S}^n = (S^n, g_{eucl}) \subset \mathbb{R}^{n+1}$ with the restr. of the metric on $\mathbb{R}^{n+1}$ to the Tg bundle to S^n. $Iso(S^n) = O(n,\mathbb{R})$
The sectional curv. is $\equiv 1$.

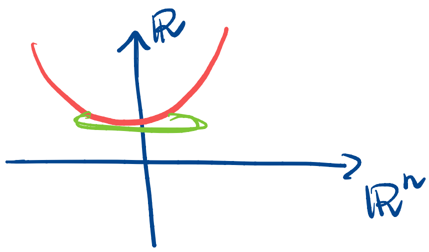
(3) Real hyp. space
$\mathcal{H}^n_{\mathbb{R}} := \{x \in \mathbb{R}^{n+1} : q(x) = 1,\ x_{n+1} > 0\}$,
where $q : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is the quadratic form associated to the symm. bilinear form of signature $(n,1)$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

$\underline{n=1}$



$\underline{n \geq 1}$



$$O(n,1) := \{g \in GL(n+1,\mathbb{R}) : q(gx) = q(x)\ \forall x \in \mathbb{R}^{n+1}\}$$

acts transitively on the two sheeted hyperboloid

$$O(n,1)_+ := \{g \in O(n,1) : g\mathcal{H}^n_{\mathbb{R}} = \mathcal{H}^n_{\mathbb{R}}\}$$

To give it a metric, write $\forall x \in \mathcal{H}^n_{\mathbb{R}}$
$$\mathbb{R}^{n+1} = \mathbb{R}x \oplus (\mathbb{R}x)^\perp$$

$$(\mathbb{R}x)^\perp := \{y \in \mathbb{R}^{n+1} : \langle x, y \rangle = 0\}$$

Since $\langle x, x \rangle = -1 \Rightarrow \langle x, x \rangle|_{(\mathbb{R}x)^\perp} >> 0$ and hence define a Riem. metric on $\mathcal{H}^n_{\mathbb{R}}$.

$Iso(\mathcal{H}^n_{\mathbb{R}}) = O(n,1)_+$ and the sectional curvature is $\equiv -1$.

Rk One could define $\mathcal{H}^n_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} = \{1, i, j, k\}$
$ij = k,\ jk = i,\ ki = j,\ i^2 = j^2 = k^2 = -1$

$$\langle x, y \rangle := \sum_{j=1}^{n} \bar{x}_j y_j - \bar{x}_{n+1} y_{n+1}$$

$$P(\mathbb{K}^{n+1}) = (\mathbb{K}^{n+1} \setminus \{0\}) / \mathbb{K}^*$$

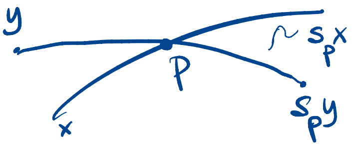
$$\mathcal{H}^n_{\mathbb{K}} := \{x \in P(\mathbb{K}^{n+1}) : q(x) < 0\}$$

kn-dim. mfd, $k = \dim_{\mathbb{R}} \mathbb{K}$.
The sectional curv. is constant $\Leftrightarrow$ $\mathbb{K} = \mathbb{R}$ (otherwise pinched negative).

II. Riem. charact.

All Riem. mfds are connected, 2nd count, paracompact, Hausdorff & finite dim. (except Lie gps)

Defn M Riem. mfd. A *geodesic symmetry* at $p \in M$ is a map s_p defn. in a nbd of p that fixes (only) p and reverses local geodesics through p.



Defn (1) The Riem. mfd. M is *locally symmetric* if $\forall p \in M$ $\exists$ geod. symmetry s_p that is an isometry on its domain

(2) The Riem. mfd M is *globally symmetric* if $\forall p \in M$ there is a globally defined geodesic symmetry.

Examples

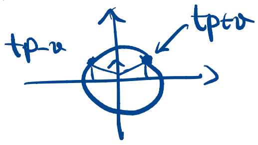
(1) $M = \mathbb{E}^n$, $p \in M \Rightarrow s_p(x) := 2p - x$

(2) $M = \underset{x \in}{S^n}$, $\mathbb{R}^{n+1} = \mathbb{R}p \oplus (\mathbb{R}p)^\perp$

$S^n = \{tp + v : \|tp+v\| = 1\}$

$s_p : S^n \to S^n$

$(tp+v) \mapsto tp - v$



(3) $M = \mathcal{H}^n_{\mathbb{R}}$, $s_p(x) = -2x\langle p,x\rangle - x$

$s_p(p) = p$, $s_p^2 = 1$

and s_p preserves the Riem. metric since $q(s_p(x)) = q(x)$

Thm (M^n, g) complete simply connected mfd with constant ~~negative~~ sectional curv., $K = \{-1, 0, 1\} \Rightarrow$

$\Rightarrow$ $K = 1$ $\quad M = S^n \quad n \geq 2$

$K = 0 \quad M = \mathbb{E}^n$

$K = -1 \quad M = \mathcal{H}^n_{\mathbb{R}}$

Properties of a Riem. glob. symm. space M:

(1) complete

(2) $\mathrm{Iso}(M)^\circ$ is finite dim.

(3) $\mathrm{Iso}(M)^\circ \curvearrowright M$ transitive

(4) $\mathrm{Stab}_{\mathrm{Iso}(M)}(p)$ is cpt $\forall p$

$\Rightarrow M \cong \mathrm{Iso}(M)^\circ / \mathrm{Stab}_{\mathrm{Iso}(M)}(p)$

Further examples

(1) Any cpt semisimple Lie gp can be turned into a Riem. symm. space

(2) Cpt connected oriented surfaces

$\underline{g=0}$ $\quad S^2$ globally symm.

$\underline{g=1}$ $\quad T^2 \simeq \mathbb{R}^2/\Gamma$, $\Gamma < \mathbb{R}^2$ discrete gp of isometries acting on $\mathbb{R}^2$ prop. disc. & with no fixed pt.

$\underline{g \geq 2}$ $\quad \Sigma \simeq \mathcal{H}^2_{\mathbb{R}}/\Gamma$, where $\Gamma < \mathrm{Iso}(\mathcal{H}^2_{\mathbb{R}}) = O(2,1)_+$ is a discrete cocpt. subgp ($\simeq \pi_1(\Sigma)$) acting on $\mathcal{H}^2_{\mathbb{R}}$ p.d. & with no f.p.

(3) $\mathcal{H}^2_{\mathbb{R}}/\Gamma$, $\Gamma = SL(2,\mathbb{Z})$, or a finite index sbgp. of $SL(2,\mathbb{Z})$ $\Rightarrow \Sigma = \mathcal{H}^2_{\mathbb{R}}/\Gamma$ is only finite volume.

(4) (A. Borel) Any Riem. symm. sp. whose isom. gp. is semisimple admits a quotient that is of finite volume and a quotient that is cpt.

$\Gamma < \mathrm{Iso}(\Sigma)$ are lattices (i.e. discrete & $\exists$ finite inv. m on $\mathrm{Iso}(\Sigma)/\Gamma$)

(5) $\mathcal{P}'(n) := \{ A \in M_{n,n}(\mathbb{R}) : \det A = 1,$ A symm. & pos. defin.$\}$

$SL(n,\mathbb{R})$ acts on $\mathcal{P}'(n)$ via $g_* A := g A {}^t g$ and the action is transitive. Moreover

$g A {}^t g = A \iff g \in SO(n,\mathbb{R})$ and

$\mathrm{Iso}(\mathcal{P}'(n)) = PSL(n,\mathbb{R}) = SL(n,\mathbb{R}) / \{\pm Id\}$

III. Algebraic charact.

G conn. Lie gp, $\sigma : G \to G$ involutive auto, $\sigma \in \mathrm{Aut}(G)$, $\sigma^2 = Id$. A symm. space for G is a homog. space G/H, where $H < G^\sigma$ is an open subgroup.

If $G^\sigma = \{ g \in G : \sigma(g) = g \}$ is cpt $\Rightarrow G^\sigma$ closed hence a Lie subgp of G and G/G^σ can be endowed with a Riem. metric. Thus for $(G^\sigma)^\circ < K < G^\sigma \Rightarrow G/K$ is a Riem. symm. space.

Ex $\mathcal{P}'(n) = PSL(n,\mathbb{R}) / SO(n)$

Rk $\sigma^2 = Id \Rightarrow d_e\sigma$ has e. values $+1, -1$. $d_e \sigma : \mathfrak{g} \to \mathfrak{g}$

$\Rightarrow$ can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

$(*)$ $\begin{cases} \mathfrak{k} = +1 \text{ eigenspace} \\ \mathfrak{p} = -1 \text{ eigenspace} \\ [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \ [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \ [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \\ \mathfrak{k} \text{ is a Lie subalgebra} \\ \mathfrak{p} \text{ is a subspace.} \end{cases}$

orthogonal symm. Lie algebras = $(\mathfrak{g}, \theta)$, $\mathfrak{g}$ Lie alg., θ inv. auto with $(*)$

IV Equiv. between the two charact.

- $M = G/K$ globally Riem. symm. space (in particular have s_p)

Define $\sigma : G \to G$ by $\sigma(g) = s_p g s_p$. This is an involutive auto $\sigma^2 = Id$, $\sigma \in \mathrm{Aut}(G)$ and $(G^\sigma)^\circ < K < G^\sigma$

- If $\sigma \in \mathrm{Aut}(G)$ with $\sigma^2 = Id$, let $(G^\sigma)^\circ < K < G^\sigma$ and define $s_p : M \to M = G/K$, for $p = hK$

$s_p(\ell K) := h\,\sigma(h^{-1}\ell)K$.

One can verify that $s_p^2 = Id$, $s_p \in \mathrm{Iso}(M)$, $s_p(p) = p$.

Rk $d_p s_p = -Id$

V Decomp. and classif.

Cartan 1926: Classif. of all s.c. Riem. symm. spaces.

- M Riem. glob. symm. $\Rightarrow$
  $M = M_0 \times M_+ \times M_-$, where
  - M_0 has zero curv. $\Rightarrow M_0 \cong \mathbb{E}^n$
  - M_+ has non-negative sect. curv.
  - M_- has non-positive sect. curv.

  M_+ are of compact type
  M_- are of non-cpt. type
  $\exists$ a duality between cpt type & non-cpt type.

look only at M_-

- rk of a symm. space =
  = dim. a totally geodesic subspace of M (i.e. the max. subsp. of tg space where curvature vanishes)

  i.e. there = dim. of a maxim. Abelian sub. that is diagonalizable

  If rk is one $\Rightarrow$ the curvature is either positive (S^n) or negative ($\mathbb{H}^n_{\mathbb{R}}$)
- various decomp (Iwasawa, Cartan, ...)
- geometry at infinity