Lecture

25 Feb 2021

II. Generalities on symm. spaces

II.1 Isometries and the isometry group

Riemannian metric on a smooth mfd M is a map g that $\forall x \in M$ associates a scalar product on T_xM with the following property: $\forall$ coord. chart $\varphi : \underset{M}{\overset{\cap}{U}} \to \mathbb{R}^n$, $1 \le i,j \le n$ the function

$$\begin{aligned} U &\longrightarrow \mathbb{R} \\ x &\longmapsto g_x\left((d_x\varphi)^{-1}(e_i), (d_x\varphi)^{-1}(e_j)\right) \end{aligned}$$

is smooth.

The **length** $l(c)$ of a smooth path $c : [a,b] \to M$ is

$$l(c) := \int_a^b \sqrt{g_{c(t)}(\dot c(t), \dot c(t))}\, dt$$

where $\dot c(t) \in T_{c(t)}M$ is the tg vector to c at $c(t)$.

If M is connected

$$d(x,y) := \inf\{\, l(c) : c : [a,b] \to M \text{ smooth path } c(a)=x,\ c(b)=y \,\}$$

defines the **Riemannian distance**

Defn Let (M,g), (N,h) be two Riem. mfds. A **Riemannian isometry** is a diffeo $f : M \to N$ s.t. $f^*h = g$, namely $\forall p \in M$, $\forall u,v \in T_pM$

$$h_{f(p)}(d_pf(u), d_pf(v)) = g_p(u,v)$$

Fact: A Riem. isometry preserves the Riem. distance.

Thm II.1 (Helgason I.11.1) Let M be a Riem. mfd and $d : M \times M \to [0,\infty)$ the Riem. distance. If $\varphi : M \to M$ is a diffeo, TFAE:

(1) φ Riem. isom.

(2) φ distance preserving bijection.

Lemma II.2 (Helgason I.12.2) Let $f_i : M \to N$, $i=1,2$ two isom. between Riem. mfds and assume that M is connected. Suppose that for some $p \in M$

$f_1(p) = f_2(p)$

$d_pf_1 = d_pf_2$

Then $f_1 = f_2$

Recall The **Riemannian expon. map** $\mathrm{Exp}_p : T_pM \to M$ is defined by $\mathrm{Exp}_p(X_p) := \gamma_{X_p}(1)$,

where $\gamma_{X_p} : (-2,2) \to M$ is the unique geodesic s.t.

$$\gamma_{X_p}(0) = p, \quad \dot\gamma_{X_p}(0) = X_p.$$

A **normal neighborhood** of p is an open nbd of p that is the diffeomorphic image of a star shaped nbd of $0 \in T_pM$, under Exp_p.

If $f : M \to M$ is an isometry,

$$\begin{tikzcd} T_pM \arrow[r, "d_pf"] \arrow[d, "\mathrm{Exp}_p"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{f(p)}M \arrow[d, "\mathrm{Exp}_{f(p)}"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

Proof of lemma II.2 $f := f_2^{-1}\circ f_1 : M \to M$ is an isometry that satisfies

$$\begin{cases} f(p)=p \\ d_pf = Id \end{cases}$$

$S := \{q\in M : f(q)=q,\ d_qf = Id\} \ni p$ is closed and not empty.

Let $q\in S$ and $U = Exp_q(N_0)$ a normal neighborhood. $\Rightarrow$

$\forall\, v\in T_qM$ and $t\in\mathbb{R}$ with $tv\in N_0$

$$f(Exp_q(tv)) = Exp_{f(q)}(t\, d_q f(v)) = Exp_q(t\, \underbrace{d_qf}_{Id}(v)) = Exp_q(tv)$$

$\Rightarrow f|_U = Id \Rightarrow U\subset S$. ∎

$Iso(M)$ = Isom. gp of M with the compact-open topology, i.e. the top. with the subbasis of open sets

$$S(C,U) := \{f\in Iso(M) : f(C)\subset U\}$$

where $C\subset M$ is cpt, $U\subset M$ open.

Theorem II.3 $Iso(M)$ with the cpt open top is a locally cpt gp acting continuously on M.
Moreover $Stab_{Iso(M)}(p)$ is compact.

Idea of the proof It relies upon:

(1) M metric space $\Rightarrow$ cpt open top on $Iso(M)$ = top. of unif. conv. on cpt sets.

(2) If $(f_n)_{n\geq 1}\subset M$ is a sequence such that for some $p_0\in M$, $(f_n(p_0))_{n\geq 1}$ converges $\Rightarrow$ $\exists$ a convergent subsequence
(Helgason IV 2.2, uses H IV 2.3, IV 2.4)

To see the cpt. of stabil. consider

$$Stab_{Iso(M)}(p) \longrightarrow O(T_pM)$$
$$f \longmapsto d_pf$$

If $d_pf = Id \Rightarrow f = id$ ∎

Geodesic symmetries.

Defn. M a Riem. mfd. $p\in M$.

(a) M is Riemannian locally symmetric if for every $p\in M$ there exist a normal nbd $U\ni p$ and an isometry $s_p : U\to U$ s.t.
(1) $s_p^2 = Id$
(2) p is the only fixed pt of s_p in U.

s_p is a geodesic symmetry

(b) M is Riem. globally symm. if $\forall\, p\in M$ s_p can be extended to M.

Thm II.4 (Helgason IV.5.6)
A complete simply conn. Riem. locally symm. space is Riem. globally symm. In particular the univ. cov. of a locally symm. space is globally symm. and every loc. symm. space is a quotient of a globally symm. space by a discrete torsion-free gp of isometries isom. to the fund. gp.

Lemma II.5 M Riem. mfd, $p\in U\subset M$, $U = Exp(N_0)$ is a normal nbd of p. Let $s_p\in Iso(M)$ be an isometry s.t. p is the only fixed pt.

Then TFAE:

(1) $s_p^2 = id$

(2) $d_p s_p = -id$

In either case then

$$s_p(\mathrm{Exp}_p(tv)) = \mathrm{Exp}_p(-tv)$$

wherever Exp is defined.

Pf (2) ⇒ (1) $d_p s_p = -id$ ⇒

⇒ $d_p (s_p)^2 = Id$ ⇒ lemma II.2

(1) ⇒ (2) $s_p^2 = id$ ⇒

⇒ $(d_p s_p)^2 = Id$ ⇒ $d_p s_p$ has e.values +1 or −1. If $d_p s_p$ were to have e.v. +1 ⇒

⇒ ∃ $v \in T_pM$ with $d_p s_p(v) = v$.

Thus ∀ $tv \in N_0$

$$s_p(\mathrm{Exp}\, tv) = \mathrm{Exp}(d_p s_p(tv)) =$$

$= \mathrm{Exp}(tv)$ ⇒ $\mathrm{Exp}(tv)$ would be a fixed pt ∀t s.t. $tv \in N_0$ ↯.

Corollary II.6 M conn. Riem. mfd, $p \in M$. Then ∃ at most one involutive isometry of M with p as isolated fixed pt.

Proposition II.7 M Riem. symm. space. Then M is complete (as metric space). Moreover $\mathrm{Iso}(M)^{\circ}$ acts transitively.

Hopf-Rinow theorem M conn. Riem.

TFAE:

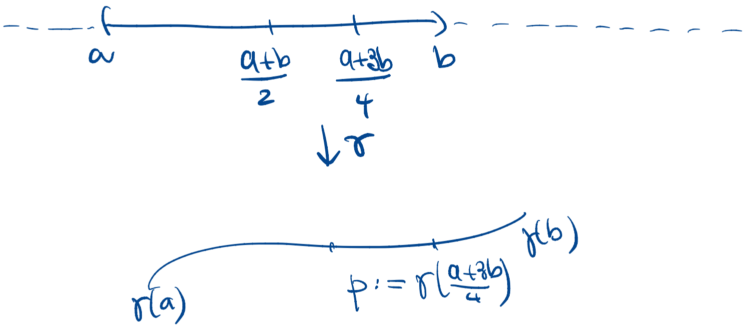
(1) Closed & bdd sets are compact

(2) M is a complete metric space

(3) M is geod. complete, that is $\forall p \in M$ the exp. map is defn. on the whole T_p space.

As a conseq. of any of the above, $\forall p, q \in M$ ∃ geod. conn. p, q.

Pf Let $\gamma : (a,b) \to M$ be a geodesic segment. We'll show that γ can be extended to $(a, b + \frac{b-a}{2})$. This will show that γ can be extended to $\mathbb{R}$ hence, by Hopf-Rinow, M is complete.



Let

$$\eta : \left(\frac{a+b}{2}, b + \frac{b-a}{2}\right) \longrightarrow M \text{ be the geod.}$$

$$t \longmapsto s_p\left(\gamma\left(\frac{a+3b}{2} - t\right)\right)$$

$$a < \frac{a+3b}{2} - t < b \quad \text{if}$$

$$t \in \left(\frac{a+b}{2}, b + \frac{b-a}{2}\right)$$

Then $\eta\left(\frac{a+3b}{4}\right) = \gamma\left(\frac{a+3b}{4}\right) = p$

since s_p fixes p

Also

$$\dot\eta\left(\frac{a+3b}{4}\right) = \frac{d}{dt}\left(s_p \gamma\left(\frac{a+3b}{4} - t\right)\right) =$$

$$= d_p s_p\left(-\dot\gamma\left(\frac{a+3b}{4}\right)\right) =$$

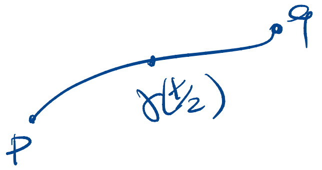
$$= \dot\gamma\left(\frac{a+3b}{4}\right) \Rightarrow$$

$$\Rightarrow \eta\big|_{\left(\frac{a+b}{2}, b\right)} = \gamma\big|_{\left(\frac{a+b}{2}, b\right)}$$

⇒ we can prolong γ to $(a, b + \frac{b-a}{2})$ using η.

Let $p, q \in M$ and $\gamma : [0,t] \to M$ with $\gamma(0) = p$, $\gamma(t) = q$. Then

$$q = s_{\gamma(t/2)}(p)$$



To see that $Iso(M)^\circ$ (not only $Iso(M)$) acts transitively need

Lemma II.8 Let M be a symm. space. Then the map

$$M \longrightarrow Iso(M)$$
$$p \longmapsto s_p$$

is continuous.

Pf Next week

$\Rightarrow$ the geod. symm. are is some conn. cpt. of $Iso(M)$. We want to show that they are in $Iso(M)^\circ$.

$$M \times M \longrightarrow Iso(M)$$
$$(p,q) \longrightarrow s_p \circ s_q$$

This is continuous and the image contains $s_p^2 = id$

Is this enough ?